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Markov transition matrices and Fibonacci numbers

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Markov Transition Matrices And Fibonacci Numbers

by

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Abstract

In this paper, we examine various classes of Markov transition matrices whose limiting probabilities is the normalized sequence of Fibonacci numbers. We investigate the various properties of these classes of matrices. In addition, general results about the class of transition matrices whose limiting probability vectors form other known sequences are presented.

Dedication

To my parents, Albert and Abigail Olu-Sajobi, who taught me to stay true to my beliefs despite all odds and become the best I can ever be.

To the millions of brilliant minds in the third world countries, trapped in the struggle for the basic necessities of life, unable to maximize their potentials.

Acknowledgements

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To my loving parents, thank you for giving me a good platform to achieve my goals. Your unflinching support and teachings have spurred me on to follow my passions. Many thanks to my siblings, Taiwo, Kehinde, and Ireoluwa, thank you for your unwavering support for "Bros T", you guys are simply the best. To my Abimbola Popoola, thank you for allowing me to go this far. Your love, devotion and encouragement always challenges me to success.

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CHAPTER 1

Introduction

Markov processes have been found to be appropriate in modelling many problems arising in nature. In many of these problems, of great importance to analysts is the stationary distribution of the Markov processes modelled. In fact, most control decisions are based on the stationary distribution of the processes. As a result, several methods have been developed to solve for the stationary distribution of systems that are modelled by Markov processes. Some of the methods include the generating function approach (see Bhat and Miller [**1**]), the numerical approach, the uniformization approach (for Continuous Time Markov Processes (CTMP)), and so on.

Fibonacci numbers form a sequence of positive integers whose terms are sum of the two preceding numbers given two initial numbers. Research on Fibonacci numbers has been going on since the 13th century when the numbers were found by Leonardo Fibonacci. Interesting properties of Fibonacci numbers have been found, and many applications have be discovered. Fibonacci numbers have been found useful in many areas of human endeavors like Architecture, Human Anatomy, Dentistry, Computer sciences, investment management, Music, and so on.

The literature abounds with papers on number theory (especially number sequences) using matrix methods (but not probabilistic matrices). Basin and Hoggatt, in their novel paper [**2**], presented a matrix method approach for analyzing properties of Fibonacci numbers. This gave rise to a lot of papers on the Fibonacci Q-matrix method of finding properties of Fibonacci numbers. Since then, matrix methods have been employed in finding interesting properties of number sequences. Kalman [**3**] derived a number of closed-form formulas for the generalized Fibonacci sequence by matrix methods. Er [4] extended the matrix representation and showed that sums of generalized Fibonacci numbers could be derived directly using this representation. However, the literature is scanty on research into the application of number theory to probability (especially to the analysis of Markov processes). One of the few works in this area was by Simeon Berman [5]. He considered a certain discrete time Markov process in the interval (0,1] and defined the process X_n to have a binary expansion. He then found properties of the process.

In this study, we use the given stationary distribution (including the infinite state space case) and find probability transition matrices (or the rate matrices) with the given stationary distribution for some classes of Markov processes. We examine some Markov processes whose limiting probability vectors form known sequences of numbers. The properties of such known sequences are applied in simplifying the limiting probability vector for the underlying Markov processes. The basic idea of this work is to use knowledge of number sequences in constructing classes of transition matrices. Also using the properties of number sequences (especially Fibonacci numbers), the limiting probability vector for the finite and infinite classes of these transition matrices are analyzed

In the next section, we present some definition of the terms used in this thesis. In chapter 2, the concept of Uniformization is presented and a geometric interpretation of uniformized Markov processes is given. In chapters 3 and 4, interesting results about the class of transition matrices, whose limiting probability vector is the normalized sequence of Fibonacci numbers, are presented. The class of transition matrices whose limiting probability vectors form normalized sequences belonging to the class of generalized Fibonacci sequences are presented in chapter 5. In Chapter 6, we examine some other types of sequences as examples. General results for classes of transition matrices whose limiting probability vectors are normalized Horadam sequences are presented in chapter 7. Two new classes of transition matrices, whose limiting probability vectors are normalized Horadam sequences, are presented in chapter 8. Some common properties of the classes of transition matrices considered are also discussed. Finally in chapter 9, we present some concluding comments and outline issues for future research.

1.1. Definitions

We define some of the terms used in the thesis.

Markov Processes

A discrete-parameter stochastic process $\{X(t), t = 0, 1, 2, ...\}$ or a continuous-parameter stochastic process $\{X(t), t > 0\}$ is said to be a Markov process if for any set n time points $t_1 < t_2 < ... < t_n$ in the index set or time range of the process, the conditional distribution of $X(t)$ being at state n given the values of $X(t_1), X(t_2), X(t_3), \ldots, X(t_{n-1})$, depends on the immediately preceding value $X(t_{n-1})$. More precisely [**11**], for any real numbers $x_1, x_2, ..., x_n$

$$Pr\{X(t_n) \leq x_n | X(t_1) = x_1, ..., X(t_{n-1}) = x_{n-1}\} = Pr\{X(t_n) \leq x_n | X(t_{n-1}) = x_{n-1}\}.$$

That is, given the most recent data, the future is independent of the past data. Many real life processes can be model by Markov processes. Examples include reliability processes, queueing systems, computer networks, and so on.

Markov Chain

This is a class of Markov processes where the state space and the time index are both discrete. When the time index is continuous, the chain is said to be a continuous-time Markov chain. An example of this is birth and death processes. When the state space is continuous, we have a Markov process.

Transition Probability Matrix

Transition Probability matrix is the matrix of the probabilities of one-step transition from state i to j. Therefore, the entries of the matrix are probabilities and each rows add up to 1. It is usually denoted by $P = [p_{ij}]$, where $p_{ij} = P(X_{n+1} = j|X_n = i)$. For example,

$$P=\begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Limiting Distribution

The limiting probability is the probability that after a long time, the process is in state j given that it started in state i and it is independent of the starting state i. In other words, the limiting distribution describes the long run behavior of the Markov process. The limiting distribution[**11**] is usually represented by a row vector(say ν) with the condition that the entries sum to 1. Given a transition matrix P,

$$lim_{n\to\infty} P^n = \nu.$$

Irreducibility

A Markov chain is said to be irreducible if the states of the process communicate. That is, if it is possible to get to any state from any state.

Periodicity

The period of a state i is defined as the greatest common divisor of all integers $n \geq 1$ for which $P_{ii}^{(n)} > 0$ [**1**]. That is, a state has period k if any return to state i can occur only in some multiple of k time steps and k is the largest number with this property. In other words, the period of a state is defined as

$$k = gcd\{n : Pr(X_n = j|X_0 = i) > 0\}$$

When the period is 1, then the state is said to be aperiodic; otherwise, the state is said to be periodic with period k.

Recurrence

A state i is said to be recurrent if starting from state i, eventual return to this state is certain[**1**]. For a given state i, the mean recurrence time μ_i is defined as

$$\mu_i = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$

A recurrent state i is said to be positive recurrent if $\mu_i < \infty$. Otherwise, it is said to be null recurrent.

Ergodicity

A state i is said to be ergodic if it is aperiodic and positive recurrent. If all states in a Markov chain are ergodic, then the chain is said to be ergodic.

Rate Matrix

Let μ_i be the rate at which the process leaves state i (that is, the average number of "leavings" per unit time given that the system is in state i). Let p_{ij} be the probability that the process goes from state i directly to state j. Define $q_{ij} = \mu_i p_{ij}$ for all $i \neq j$. Then, q_{ij} is called the transition rate from i to j.

The rate matrix or infinitesimal generator (often denoted by Q) is the matrix of the transition rates from state i to j. Each row sums to zero and all the diagonal elements are negative. That is, q_{ii} is defined to be $\sum_j q_{ij}$, where $i \neq j$.

1.2. Contributions

In this thesis, we find transition and rate matrices that have a given limiting distribution consisting of well known normalized sequences in number theory. The results given are all new and original, except for some of the material in the uniformization chapter.

CHAPTER 2

Uniformization

Uniformization (also known as Randomization) was introduced by Jensen [**6**], popularized and expanded by Grassmann [**7**] and has been used by many other authors. Its underlying goal is to compute the transient probability vector of a continuous time Markov chain by converting the process to a discrete time Markov chain (DTMC) and solving for the probability vector in that setting. We use the tools of Uniformization for a different purpose.

We consider an ergodic continuous time Markov chain $X(t) = \{X(t), t \in [0, n)\}$ with rate matrix (infinitesimal generator) Q and the limiting probability (row) vector ν. While many authors employ the uniformization approach to find a DTMC whose transient probability vector is the same as for CTMC, we use this method to obtain a class of transition matrices with the same stationary distribution as the Q rate matrix.

THEOREM 2.0.1.

Consider an ergodic continuous time Markov chain with the rate matrix Q. Let ν be the limiting probability row vector for this Markov process.

(a) *Then, $I + (Q/k)$ is a probability transition matrix with the same limiting probability vector ν if $k \geq max_{i,j}|Q_{i,j}|$, where k is a real number and Q_{ij} is the (i, j) entry of Q.*

(b) *If P is a probability transition matrix, then $P - I$ is a rate matrix.*

PROOF.

(a) The standard method of obtaining the limiting vector for a given rate matrix Q is to solve $\underline{0} = \nu Q$. The standard method for obtaining the limiting vector for a given Markov transition matrix P is to solve $\nu = \nu P$. By Uniformization,

$$\underline{0} = \nu(Q/k).$$

Thus

$$\nu = \nu + \nu(Q/k) = \nu I + \nu(Q/k)$$
$$\Longrightarrow \nu = \nu(I + Q/k).$$

Hence, $P = I + (Q/k)$ satisfies $\nu = \nu P$ and the rows of $I + (Q/k)$ sum to 1 while the entries of $I + (Q/k)$ lie between 0 and 1.

(b) Secondly,

$$\nu = \nu P \Longrightarrow \nu I = \nu P.$$

Hence,

$$0 = \nu P - \nu I = \nu(P - I).$$

Therefore, $\mathbf{Q} = \mathbf{P} - \mathbf{I}$ satisfies $\underline{0} = \nu\mathbf{Q}$ and the rows of $\mathbf{Q}$ sum to 0 while its non-diagonal entries are non-negative. □

THEOREM 2.0.2.

Let P be an $n \times n$ probability transition matrix for a DTMC with limiting probability vector ν. The class of transition matrices of the form $I + (P - I)/k$ has the same limiting vector as P, where k is a number such that $k \geq max_{i,j}|Q_{i,j}|$ and $P - I = [Q_{i,j}]$ and I is an $n \times n$ identity matrix.

PROOF.

Since $\nu = \nu P$, it follows that $\underline{0} = \nu(P - I)$. Also P-I satisfies the condition of a rate matrix of a CTMP, namely that the rows sum to 0, the diagonal entries are negative and the off diagonal entries are non-negative. We can divide the entries of $P - I$ by a real number $k > 0$ and still have a rate matrix with the same limiting probability vector. However, we want to convert our rate matrix back to a transition matrix so we need k at least as large as the largest absolute entry in $P - I$. Select any such k. Then $\underline{0} = \nu(P - I)/k$. Add $\nu = I\nu$ to both sides to get $\nu = \nu[(P - I)/k + I]$. Because of the way we chose k, we know that $\nu = \nu[(P - I)/k + I]$ satisfies the conditions of a DTMC probability transition matrix, namely that the rows sum to 1 and all the entries lie between 0 and 1. We still have the same limiting probability vector. □

Hence for any continuous time Markov chain $X(t)$ with rate matrix Q, we have a whole class of discrete time transition matrices with the same stationary distribution as Q. This affords us the opportunity of presenting continuous time processes (such as queueing systems, reliability systems and so on) in a Markov transition matrix form. Furthermore, Theorems (2.0.1) and (2.0.2) make it possible to move back and forth between the continuous time Markov chain and the discrete time Markov chains. In the subsequent pages, the continuous-time Markov chain are transformed to discrete time Markov chains with their associated probability transition matrices. We illustrate theorems 2.0.1 and 2.0.2 with the following example.

EXAMPLE 2.0.1.

We consider an irreducible, finite state aperiodic discrete time Markov chain with states $\{0, 1, 2, 3, 4\}$, *and with the transition probability matrix*

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Let ν be the limiting probability (row) vector. Solving for the limiting probability vector ν using $\nu = \nu P$, given that $\sum \nu_i = 1$, gives

$$\nu = (\nu_0, 2\nu_0 - 1, 5\nu_0 - 3, 13\nu_0 - 8, 34\nu_0 - 21).$$

$$\nu = \tfrac{1}{55}(34, 13, 5, 2, 1).$$

By Uniformization , we can generate a continuous time Markov process with a rate matrix $P - I$ with the same limiting probability vector as P,

$$Q = P - I = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}.$$

Letting $k = max_{i,j}|Q_{i,j}| = \frac{3}{4}$, we get another continuous time Markov process with the rate matrix Q^* and the same limiting vector ν, namely,

$$Q^* = \frac{Q}{k} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & -1 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}.$$

Finally, we get a transition matrix P^* which is different from P.

$$P^* = Q^* + I = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Hence for different choices of $k \geq max_{i,j}|Q_{i,j}|$, *we get a class of transition matrices with the same limiting probability vector* ν.

Solving for the limiting probability vector using $\nu = \nu P$ can be cumbersome for transition matrices of higher finite dimensions if we begin with the first column. We therefore, solve for ν starting from the last column.

2.1. Geometric Interpretation of Uniformization

Brill and Hlynka [8] found a geometric interpretation of 2×2 Markov transition matrices, by representing a transition matrix P by the (x, y) coordinates which are the first entries of the two rows of

$$P = \begin{pmatrix} x & 1-x \\ y & 1-y \end{pmatrix}$$

PROPERTY 2.1.1. *(Brill and Hlynka, 2002)*

Let $\nu = (a, 1-a)$ *be a limiting probability vector for a* 2×2 *transition matrix* P *with* $0 < a < 1$. *For* $0 < x < 1$, $0 < y < 1$, *let the transition matrix*

$$\boldsymbol{P} = \begin{pmatrix} x & 1-x \\ y & 1-y \end{pmatrix}$$

be represented by (x,y). Then the set of all transition matrices P *having* ν *as the limiting probability vector, when represented in* $\mathbb{R}^2$, *forms a straight line in the first quadrant through the points* $(1, 0)$, (a, a) *and* $(0, \frac{a}{1-a})$.

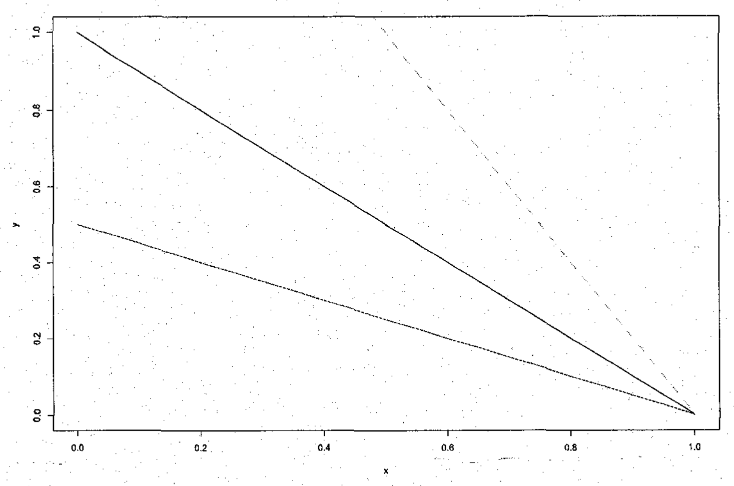


FIGURE 2.1. The Geometric representation of 2×2 transition matrices.

For example, for $a = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, the graphical representation of the class of transition matrices whose limiting probability vector is given by $\nu = (a, 1-a)$ is shown in Figure 2.1.

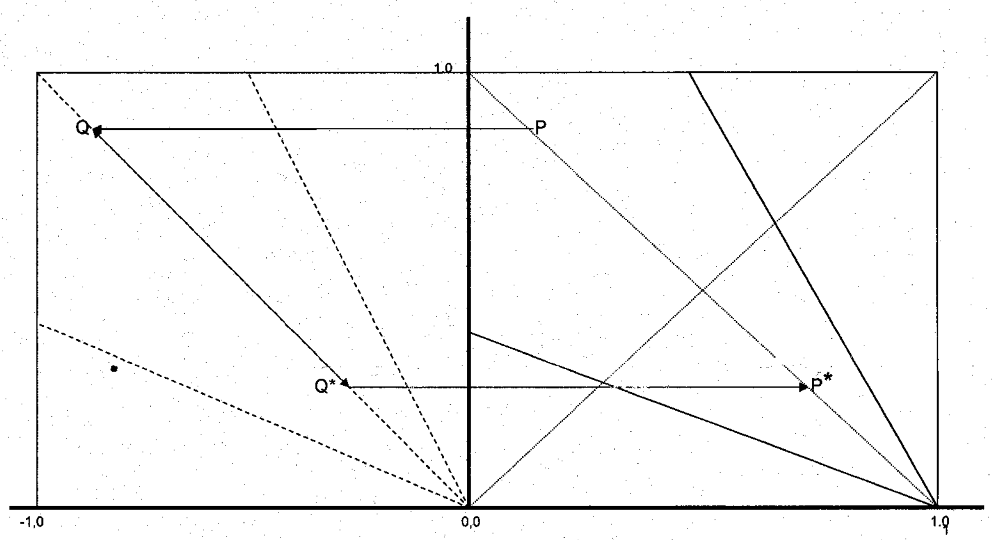


FIGURE 2.2. The Geometric representation of uniformized transition matrices.

We explain Theorem 2.0.2 by the uniformization method. Consider the geometric representation of a 2×2 transition matrix P_1 on the line joining $(1,0)$ and $(0, a/(1-a))$ in the first quadrant. By uniformization and using property 1.2, we find a corresponding rate matrix $Q = P_1 - I$ on the line $y = -(\frac{a}{1-a})x$ in the second quadrant. Dividing by $k \geq max_{i,j}|Q_{1_{i,j}}|$, another rate matrix Q^* is found. Adding the identity matrix I of the same dimension takes us to P^*, which is another transition matrix on the line $y = -\frac{a}{1-a}(x-1)$ in the first quadrant. On the other hand, starting from Q_1,we end up with a class of transition matrices on the line $y = -\frac{a}{1-a}(x-1)$ with the same stationary distribution vector as Q_1. As an example, consider the doubly stochastic transition matrix P on the line $y = 1 - x$ for an irreducible aperiodic Markov chain as shown in Figure 2.2. We then find the rate matrix $Q = P - I$ on the line $y = -x$

in the second quadrant. The matrix Q^* is also on the line $y = -x$. Adding I to Q^* gives P^* on $y = 1 - x$.

CHAPTER 3

Transition Matrices and Fibonacci Numbers

Fibonacci numbers form a sequence of positive integers governed by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}; \qquad n = 2, 3, \ldots$$

and the initial values $F_0 = 0, F_1 = 1$.

That is, after the initial values, each number is the sum of its two preceding numbers [9]. The first few Fibonacci numbers are

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377, \ldots\ .$$

In this section, we consider the class of transition matrices whose limiting probability vectors is the normalized sequences of Fibonacci numbers. By "normalized" we mean the vectors are adjusted so that the components sum to one. We further consider some properties of this class of transition matrices.

Taking a second look at Example (2.0.1), it will be noticed that the limiting probability vector forms the sequence of "every other" Fibonacci number. In that example, for states 2, 3, 4, the probabilities of moving up or moving down, staying in the same state, or moving back to state 0 are the same. Circumstances that fit the above description arise naturally. An example of this is a gambling game where the possible outcomes are winning a bet, drawing (that is, staying at the same amount of dollars), losing a bet by one dollar or losing everything(ruin). The probabilities in this kind of gambling are the same. We consider other examples of this kind by extending the 5×5 matrix to transition matrices of higher finite dimensions with the same pattern.

EXAMPLE 3.0.1.

We extend Example (2.0.1) to the 7×7 *transition matrix*

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

The limiting probability vector

$$\nu = \tfrac{1}{377}(233, 89, 34, 13, 5, 2, 1).$$

This also contains Fibonacci numbers in the limiting probability vector. By uniformization we can get a class of 7×7 transition matrices with the same limiting probability vector. By extending this to transition matrices of higher finite dimension, we have the following general result.

THEOREM 3.0.2.

For $0 < b \leq \frac{1}{3}$*, a class of* $n \times n$ *transition probability matrices that generates "every other" (odd indexed) Fibonacci number is given by*

$$\begin{pmatrix} 1-b & b & 0 & 0 & . & . & 0 & 0 & 0 \\ 2b & 1-3b & b & . & . & . & 0 & 0 & 0 \\ b & b & 1-3b & b & . & . & 0 & 0 & 0 \\ b & 0 & b & 1-3b & b & . & 0 & 0 & 0 \\ b & 0 & 0 & b & 1-3b & . & 0 & 0 & 0 \\ . & . & . & . & .. & . & . & . & . \\ . & . & . & . & .. & . & . & . & . \\ b & . & . & . & .. & . & b & 1-3b & b \\ b & 0 & 0 & 0 & 0 & . & 0 & b & 1-2b \end{pmatrix}$$

with

$$\nu = \frac{1}{\sum_{i=1}^{n} F_{2i-1}}(F_{2n-1}, F_{2n-3}, ..., F_3, F_1).$$

PROOF.

Let the limiting probability vector be

$$\nu = (\nu_0, \nu_1, \nu_2, \dots, \nu_{n-2}, \nu_{n-1}).$$

We solve $\nu = \nu P$. The balance equation for state i is

$$\nu_i = b\nu_{i-1} + (1-3b)\nu_i + b\nu_{i+1}.$$

for $i = 1, 2, \dots, n-2$. This reduces to

$$\nu_{i+1} = 3\nu_i - \nu_{i-1}. \tag{1}$$

From the definition of Fibonacci numbers,

$$F_k = F_{k-1} + F_{k-2},$$

$$F_{k-1} = F_{k-2} + F_{k-3},$$

$$F_{k-3} = F_{k-2} - F_{k-4}.$$

Hence, summing this equations gives

$$F_k = 3F_{k-2} - F_{k-4}. \tag{2}$$

The Fibonacci numbers are in reverse order in (2) and use every second subscript so we need to:

(i) reverse the order of the recursion
(ii) increase the subscript spread from 1 to 2

We accomplish (i) by

(a) Subtract middle subscript from each subscript . Thus

$$\nu_{i+1} = 3\nu_i - \nu_{i-1}$$

becomes (using the letter a)

$$a_1 = 3a_0 - a_{-1}.$$

(b) Replace each subscript by its negative

$$a_{-1} = 3a_0 - a_1.$$

(c) Shift by adding j to each subscript.

$$a_{j-1} = 3a_j - a_{j+1}$$

We accomplish (ii) by doubling each subscript

$$a_{2j-2} = 3a_{2j} - a_{2j+2}$$

Then replace $2j$ by k. Change a to F. Hence we must prove

$$F_{k-2} = 3F_k - F_{k+2}.$$

This matches (2). The initial values of the sequence is accounted for in the last column of the matrix, so the result follows. □

We next examine the properties of this class of matrices under consideration. It is well known that for every other Fibonacci number [9]

$$F_1 + F_3 + \cdots + F_{2n-3} + F_{2n-1} = F_{2n}.$$

Hence we have the following corollary.

COROLLARY 3.0.3.

For the conditions of Theorem 3.0.2, ν can be simplified to

$$\nu = \tfrac{1}{F_{2n}}(F_{2n-1}, F_{2n-3}, \ldots F_3, F_1).$$

Geometrically, the graph of the limiting probabilities versus the states is shown in Figure 3.1 for the 10×10 transition matrix case.

For the transition matrix P given in Theorem 3.0.2, we can get a new transition matrix P^1 whose limiting probability vector elements are an increasing sequence of Fibonacci numbers. Reflect the transition matrix P through its center by $p_{ij} \Leftrightarrow p_{n+1-i,n+1-j}$ to get P'. In other words, interchange the first and the last column of the transition matrix. Then flip the interchanged columns. Do the same with each pair of columns at the two ends. This implies that by transforming P to P', we can obtain the limiting probability vector ν such that an increasing sequence of every other Fibonacci numbers is obtained as the state number increases.

THEOREM 3.0.4. *(Transformation Property)*

For $0 < b \leq \frac{1}{3}$, the class transition matrices P^1 of the form

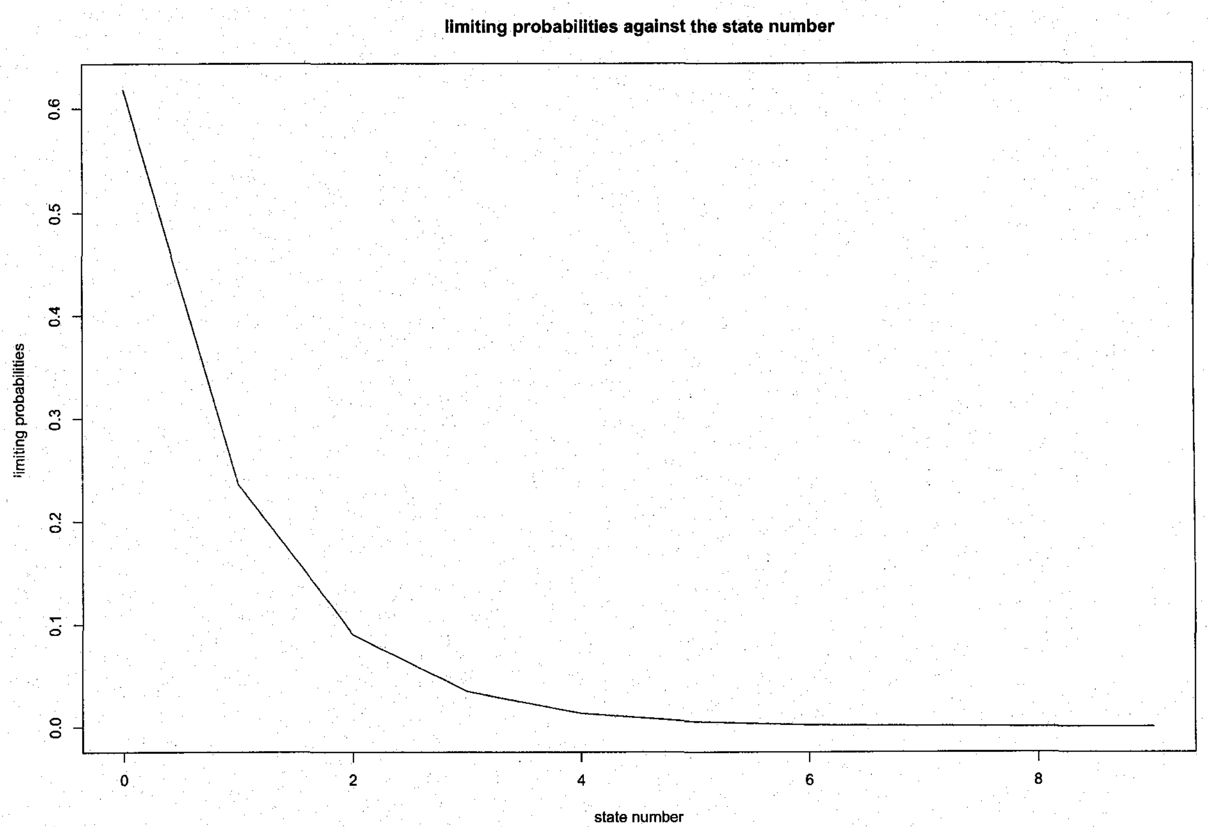


FIGURE 3.1. The Graph of limiting probabilities against the state number.

$$\begin{pmatrix}
1-2b & b & 0 & 0 & . & . & . & . & b \\
b & 1-3b & b & . & . & . & . & . & b \\
0 & b & 1-3b & b & . & . & . & . & b \\
0 & 0 & b & 1-3b & b & . & . & . & b \\
0 & 0 & 0 & b & 1-3b & . & . & . & . \\
. & . & . & . & .. & . & . & . & . \\
. & . & . & . & .. & . & . & . & . \\
. & . & . & . & .. & . & b & 1-3b & 2b \\
0 & . & . & . & . & . & . & b & 1-b
\end{pmatrix}$$

has limiting vector given as

$$\nu = \frac{1}{F_{2n}}(F_1, F_3, \ldots.., F_{2n-3}, F_{2n-1}).$$

The class of transition matrices P' represents systems in which the probability of having an empty system is very low. For example, a queueing system in which the limiting probability vector is given as shown above will frequently have a crowded (busy) system. On the other hand, the class of transition matrices P will frequently have an empty system. Customers will prefer patronizing systems with the transition matrix P as it reduces their waiting times.

However, profit-oriented business owners will prefer a system with the transition matrix P' as it implies more profit for them. As will be seen later, this transformational property is applicable to all the classes of transition matrices considered in this work. A more general result on this property of the transition matrices is presented in chapter 8. Meanwhile, we illustrate the above result with an example.

EXAMPLE 3.0.2.

For $0 < b \leq \frac{1}{3}$, consider a 5×5 transition matrix from Theorem 3.0.2, the corresponding transformed class of transition matrices given by

$$\begin{pmatrix} 1-2b & b & 0 & 0 & b \\ b & 1-3b & b & 0 & b \\ 0 & b & 1-3b & b & b \\ 0 & 0 & b & 1-3b & b \\ 0 & 0 & 0 & b & 1-b \end{pmatrix}$$

has the limiting probability vector

$$\nu = \tfrac{1}{55}(1, 2, 5, 13, 34).$$

Figure 3.2 shows the graphical representation of the limiting probabilities for the corresponding class of 10×10 transition matrices.

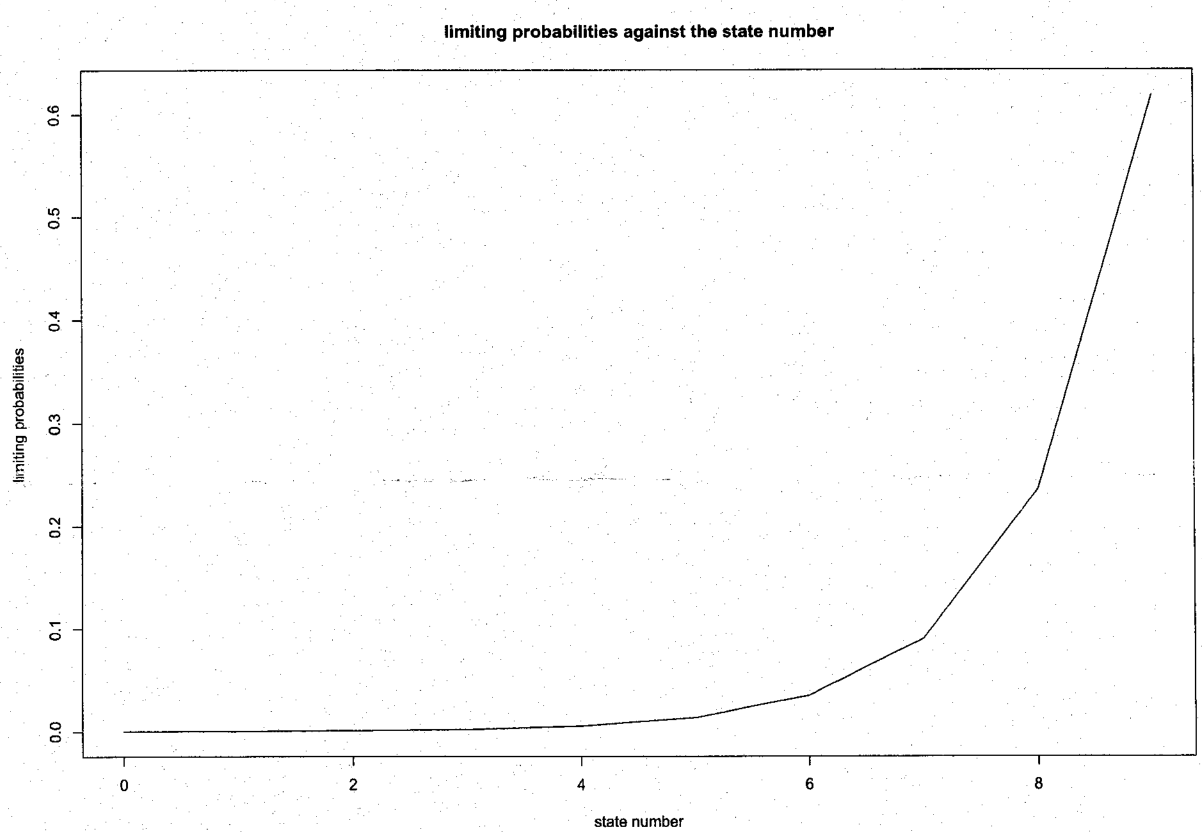


FIGURE 3.2. The graph of limiting probabilities against the state number.

DEFINITION 3.0.5. *The Fibonacci's golden ratio which is* $lim_{n\to\infty}\frac{F_{n+1}}{F_n}$ *is the unique positive real root* [9] *of*

$$x^2 - x - 1 = 0.$$

DEFINITION 3.0.6. *The Fibonacci numbers can be expressed in terms of* n*,*

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \tag{3}$$

This is also known as the Binet formula. [9]

THEOREM 3.0.7. *(Infinite State Property)*

For $0 < b \leq \frac{1}{3}$*, the infinite state space transition matrix*

$$P = \begin{pmatrix} 1-b & b & 0 & . & . & . & . & . \\ 2b & 1-3b & b & 0 & . & . & . & . \\ b & b & 1-3b & b & 0 & . & . & . \\ b & 0 & b & 1-3b & b & 0 & . & . \\ b & 0 & 0 & b & 1-3b & b & 0 & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

has the limiting probability vector

$$\nu = (\ell, \ell^3, \ell^5, \ell^7, \ldots)$$

where $\ell = \frac{\sqrt{5}-1}{2}$.

PROOF.

Using the $\nu = \nu P$ approach, we write the equations corresponding to each column beginning from the first column. We have

$$\nu_0 = (1-b)\nu_0 + 2b\nu_1 + b\nu_2 + b\nu_3 + \ldots. \tag{4}$$

$$\nu_1 = b\nu_0 + (1-3b)\nu_1 + b\nu_2, \tag{5}$$

$$\nu_2 = b\nu_1 + (1-3b)\nu_2 + b\nu_3, \tag{6}$$

$$\ldots.$$

Also since $\sum_{i=0}^{\infty} \nu_i = 1$, the first equation can be written in terms of ν_0 and ν_1 only. Solving, the limiting probability vector becomes

$$\nu = (\nu_0, 2\nu_0 - 1, 5\nu_0 - 3, 13\nu_0 - 8, \ldots).$$

which is still in terms of ν_0, which is yet to be determined.

Another standard method is to use the generating functions. If we define $\phi(z) = \sum_{i=0}^{\infty} \nu_i z^i$, then we can obtain an expression for $\phi(z)$ by multiplying (4) by z^0, (5) by z^1, (6) by z^2, and so on. Summing both sides and solving for $\phi(z)$ yields

$$\phi(z) = \tfrac{z-\nu_0(1-z)}{z-(1-z)^2}.$$

This still leaves the difficulty of solving for ν_0.

Using the finite case of Theorem 3.0.2, we have

$$\nu = (lim_{n\to\infty}\frac{F_{2n-1}}{F_{2n}}, lim_{n\to\infty}\frac{F_{2n-3}}{F_{2n}}, lim_{n\to\infty}\frac{F_{2n-5}}{F_{2n}}, \ldots). \qquad (7)$$

(See the note below concerning the mathematical difficulties that arise in writing this limit.)

Let $lim_{n\to\infty}\frac{F_{i-1}}{F_i} = \ell$. (We show that the limit exists in (0,1) in the appendix). Then,

$$lim_{n\to\infty}\tfrac{F_{2n-1}}{F_{2n}} = lim_{i\to\infty}\tfrac{F_{i-1}}{F_i} = \ell, \text{ and}$$

$$lim_{n\to\infty}\tfrac{F_{2n-3}}{F_{2n}} = \ell^3, \text{ and so on.}$$

Then,

$$\nu = (\ell, \ell^3, \ell^5, \ell^7, ...).$$

Hence, the components of ν forms a geometric sequence such that

$$\ell + \ell^3 + \ell^5 + ... = 1.$$

we have $\frac{\ell}{1-\ell^2} = 1$. Solving for ℓ gives

$$\ell = \tfrac{\sqrt{5}-1}{2}.$$

□

where ℓ is the reciprocal of the (Fibonacci) golden ratio.

NOTE 3.0.1.

It can be shown that the limiting vector in the infinite state space exists. It can be shown that the limits inside the expression for ν defined in (7) all exist. The form of the limiting vector for the finite space case is known. However, to make the proof technically correct, we would have to show that the expression of ν above really does give the limiting vector in the infinite space case. This does in fact happen because we have alternative methods of finding our results and the alternative methods give results that match ours, but a formal justification that the vector ν as defined in (7) is the limiting vector in the infinite case is needed. We do not have such a justification as yet. The comments here also hold for subsequent limiting results throughout this thesis.

THEOREM 3.0.8.

Let ℓ be the reciprocal of the golden ratio. Then,

$$\ell^{2n+1} = F_{2n+1}\ell - F_{2n}$$

where $\{F_i\}$ are the Fibonacci numbers.

PROOF.

Solving for ν using $\nu = \nu P$ from Theorem 3.0.7, we have

$$\nu = (\nu_0, 2\nu_0 - 1, 5\nu_0 - 3, 13\nu_0 - 8, \dots).$$

However, substituting $\nu_0 = \ell$ gives

$$\nu = (\ell, 2\ell - 1, 5\ell - 3, 13\ell - 8, \dots).$$

Comparing this to the final result of Theorem 3.0.7, we have

$$\ell^3 = 2\ell - 1;$$

The result follows. $\square$

We could also prove this result by algebraic manipulation as follows. Using Binet's formular, we have

$$F_{2n+1} = \frac{(\frac{1+\sqrt{5}}{2})^{2n+1} - (\frac{1-\sqrt{5}}{2})^{2n+1}}{\sqrt{5}},$$

and

$$F_{2n} = \frac{(\frac{1+\sqrt{5}}{2})^{2n} - (\frac{1-\sqrt{5}}{2})^{2n}}{\sqrt{5}}.$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then we have

$$F_{2n+1}\ell - F_{2n} = (\frac{\alpha^{2n+1} - \beta^{2n+1}}{\sqrt{5}})\ell - \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}}$$

$$F_{2n+1}\ell - F_{2n} = \frac{1}{\sqrt{5}}\{(\alpha^{2n+1} - \beta^{2n+1})\ell - (\alpha^{2n} - \beta^{2n})\}$$

$$F_{2n+1}\ell - F_{2n} = \frac{1}{\sqrt{5}}\{(\alpha^{2n+1}\ell - \alpha^{2n}) + (\beta^{2n} - \beta^{2n+1}\ell)\}$$

$$F_{2n+1}\ell - F_{2n} = \frac{1}{\sqrt{5}}\{\alpha^{2n}(\alpha\ell - 1) + \beta^{2n}(1 - \beta\ell)\}$$

Since ℓ is the inverse of α, we have

$$F_{2n+1}\ell - F_{2n} = \frac{1}{\sqrt{5}}\{-(\beta)^{2n}(\beta\ell - 1)\}$$

$$\beta\ell = (\frac{1-\sqrt{5}}{2})(\frac{\sqrt{5}-1}{2}) = \frac{\sqrt{5}-3}{2}$$

Hence the equation can be simplified to

$$F_{2n+1}\ell - F_{2n} = \frac{1}{\sqrt{5}}\{-(\beta)^{2n}(\frac{\sqrt{5}-3}{2} - 1)\}$$

That is,

$$F_{2n+1}\ell - F_{2n} = \{-(\beta)^{2n}(\frac{\sqrt{5}-1}{2}\} = \ell^{2n+1}$$

and the result is established.

CHAPTER 4

Limiting Vector of Every Fibonacci Number

Having found a class of matrices whose limiting vector is the normalized sequence of "every other" Fibonacci number, we consider the class of transition matrices whose limiting probability vector is the normalized sequence of every Fibonacci number.

THEOREM 4.0.9.

For $0 < b \leq \frac{1}{2}$, we consider a class of $n \times n$ transition probability matrices of the form

$$\begin{pmatrix} 1-b & b & 0 & 0 & . & 0 & 0 & 0 \\ b & 1-2b & b & 0 & . & 0 & 0 & 0 \\ b & 0 & 1-2b & b & . & 0 & 0 & 0 \\ 0 & b & 0 & 1-2b & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & 1-2b & b \\ 0 & . & . & . & . & b & 0 & 1-b \end{pmatrix}.$$

The corresponding limiting probability vector is

$$\nu = \frac{1}{\sum_{i=1}^{n} F_i}(F_n, F_{n-1}, F_{n-2}, ...F_1).$$

That is, the limiting probability vector is the normalized sequence of consecutive Fibonacci numbers.

PROOF.

Solving $\nu = \nu P$, the balance equation for state i, $i = 1, 2, \ldots, n-2$ is

$$\nu_i = b\nu_{i-1} + (1-2b)\nu_i + b\nu_{i+2}.$$

This simplifies to

$$2\nu_i = \nu_{i-1} + \nu_{i+2}. \tag{8}$$

We wish to reverse the order as shown in the proof of Theorem 3.0.2. So we must show that

$$2F_k = F_{k+1} + F_{k-2}. \tag{9}$$

From the definition of Fibonacci numbers,

$$F_{k+1} = F_k + F_{k-1},$$
$$F_{k-1} = F_k - F_{k-2},$$

Summing this to get

$$F_{k+1} = 2F_k - F_{k-2}.$$

which is the same result as (9). The initial values of the sequence is accounted for by the last column of the transition matrix. Hence the limiting probability vector forms the sequence of normalized consecutive Fibonacci numbers. □

Since it is well known for Fibonacci numbers that [**9**]

$$F_n + F_{n-1} + F_{n-2} + \ldots\ldots + F_2 + F_1 = F_{n+2} - 1,$$

we have the following corollary.

COROLLARY 4.0.10.

For the class of transition matrices given in Theorem 4.0.9, the corresponding limiting probability vector ν is

$$\nu = \tfrac{1}{F_{n+2}-1}(F_n, F_{n-1}, F_{n-2}, \ldots\ldots F_1).$$

Hence for the class of transition matrices given in theorem 4.0.9, the ratios of the consecutive components of the limiting probability vector are ratios of Fibonacci numbers. By uniformization, we find a corresponding rate matrix with the same limiting probability vector given as

$$Q = \begin{pmatrix} -b & b & 0 & 0 & . & . & . & 0 \\ b & -2b & b & 0 & . & . & . & 0 \\ b & 0 & -2b & b & . & . & . & 0 \\ 0 & b & 0 & -2b & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & -2b & b \\ 0 & 0 & 0 & . & . & b & 0 & -b \end{pmatrix}.$$

where $b > 0$. An example of this type of system is an $M/M^{[2]}/1/n-1$ queueing system with single arrivals but bulk(pair) service of size 2 (with finite buffer) with the condition that a customer is served by itself if it is the only one in the system. Another similar example is the $E_2/M/1/[\frac{n-1}{2}]$ (Erlang arrivals, exponential service) queueing system with $\lambda = \mu$ except for the second row of the transition matrix (see Kleinrock [**10**]). Where [] is the greatest integer function

Worthy of note is the fact that there are other classes of transition matrices whose limiting probability vector component ratios form Fibonacci ratios. For example, consider the rate matrix of the form

$$\begin{pmatrix} -F_{n-1} & F_{n-1} & 0 & 0 & . & . & . & 0 \\ F_n & A_n & F_{n-2} & 0 & . & . & . & 0 \\ 0 & F_{n-1} & A_{n-1} & F_{n-3} & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & F_3 & A_3 & F_1 \\ 0 & 0 & . & . & . & 0 & F_2 & -F_2 \end{pmatrix}$$

where $A_i = -F_i - F_{i-2}, \quad i = 3, 4 \dots$.

The above rate matrix is a birth and death system where the arrival and the service rates form decreasing sequences of Fibonacci numbers. The limiting probability vector component consists of a normalized sequence of Fibonacci numbers and has the same limiting probability as the matrix in Theorem 4.0.9. However, the matrix in Theorem 4.0.9 is more reasonable as a queueing system since the arrival rates and the service rates are not required to be Fibonacci sequence numbers.

THEOREM 4.0.11.

For the infinite state space probability transition matrix corresponding to Theorem 4.0.9, the corresponding steady state probability vector is given by

$$\nu = (\ell^2, \ell^3, \ell^4, ...)$$

where

$$\ell = \frac{\sqrt{5}-1}{2}.$$

PROOF.

We assume that $lim_{n\to\infty}\frac{F_n}{F_{n+2}-1}$ exists (see Appendix A).

$$\nu = (lim_{n\to\infty}\frac{F_n}{F_{n+2}-1}, lim_{n\to\infty}\frac{F_{n-1}}{F_{n+2}-1}, lim_{n\to\infty}\frac{F_{n-2}}{F_{n+2}-1}, ...)$$

so

$$\nu = (\ell^2, \ell^3, \ell^4, ...).$$

□

This implies that for the infinite state case, the probability of having an empty system is 0.382. For the infinite state space, an example of the class of transition matrices given above is the $M/M^{[2]}/1$ queueing system with infinite buffer, that is, systems with single arrival and paired service. In this system, the probability of having an empty system is lower than that of the system of Theorem (3.0.7) whose limiting probability vector forms the normalized sequence of "every other" Fibonacci number.

In the next section, we consider different classes of matrices whose limiting probabilities form Fibonacci-related sequences.

4.1. The n-step Fibonacci Sequence

The n-step Fibonacci sequences are sequences of numbers formed by adding the n preceding terms of the sequence. In this section we investigate the class of transition matrices whose limiting probability vector forms various normalized n-step Fibonacci numbers and their properties.

4.1.1. Tribonacci Sequence.

The Tribonacci sequence is the sequence of integers formed by adding the preceding three terms. The first three numbers of the sequence are defined to be

$$t_0 = 0, t_1 = 1, t_2 = 1,$$

and the sequence is governed by the relation:

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}; \quad n = 3, 4, \ldots .$$

The first few terms of the sequence are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots .$$

We present results about the class of transition matrices whose limiting probability vector forms a normialized sequence of Tribonacci numbers. Note in the following theorem that we have a matrix with two subdiagonals with almost all zero entries, as compared to Theorem 4.0.9 which had only one subdiagonal with almost all zero entries.

THEOREM 4.1.1.

For $0 < b \leq \frac{1}{2}$, consider the class of $n \times n$ transition matrices which have the same pattern as the 9×9 matrix below:

$$P = \begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 1-2b & b \\ 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 1-b \end{pmatrix}.$$

The limiting probability vector is

$$\nu = \frac{1}{\sum_{k=1}^{n} t_k}(t_n, t_{n-1}, t_{n-2}, \ldots . t_2, t_1).$$

PROOF.

For the above matrix, let

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-2}, \nu_{n-1}).$$

Solving $\nu = \nu P$ gives

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+3}, \qquad k = 1, \ldots, n-4.$$

That is,

$$2\nu_k = \nu_{k-1} + \nu_{k+3}. \tag{10}$$

Using the reversal as in the proof of Theorem 3.0.2, we must show that

$$2t_k = t_{k+1} - t_{k-3}. \tag{11}$$

From the definition of Tribonacci numbers,

$$t_k = t_{k-1} + t_{k-2} + t_{k-3},$$
$$t_{k-1} = t_{k-2} + t_{k-3} + t_{k-4},$$

so

$$t_k = 2t_{k-2} + 2t_{k-3} + t_{k-4}.$$

But $t_{k-1} - t_{k-4} = t_{k-2} + t_{k-3}$, so,

$$t_k = 2t_{k-1} - t_{k-4}.$$

Note that this equation matches equation (11). The first few terms of the sequence are contained in the last two columns. Hence the elements of the limiting probability vector are normalized Tribonacci numbers. □

DEFINITION 4.1.2.

The Tribonacci constant $\tau^* = lim_{n\to\infty}\frac{t_{k+1}}{t_k}$ *is the unique positive real root of*

$$\tau^{*3} - \tau^{*2} - \tau^* - 1 = 0.$$

That is, $\tau^* = 1.83929...$.

THEOREM 4.1.3.

For Tribonacci sequence,

(a) $lim_{n\to\infty}\frac{t_{n-1}}{t_n}$ *exists,*

(b) $lim_{n\to\infty}\frac{t_n}{\sum_{k=1}^{n} t_k}$ *exists.*

PROOF.

For Tribonacci numbers,

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}.$$

The resulting characteristic equation is given as

$$r^3 - r^2 - r - 1 = 0.$$

Let the roots of this equation be r_1, r_2, r_3. By Descartes' Rule of Signs [**18**], there exists only one unique positive real root of the equation. Assume the roots are distinct, then

$$t_n = a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n.$$

Let r_1 have the largest absolute value of the roots of the equation.

$$\frac{t_{n-1}}{t_n} = \frac{a_1{r_1}^{n-1} + a_2{r_2}^{n-1} + a_3{r_3}^{n-1}}{a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n}.$$

(a) Therefore,

$$\frac{t_{n-1}}{t_n} = \frac{1 + (\frac{a_2}{a_1})(\frac{r_2}{r_1})^{n-1} + (\frac{a_3}{a_1})(\frac{r_3}{r_1})^{n-1}}{r_1 + r_2(\frac{a_2}{a_1})(\frac{r_2}{r_1})^{n-1} + r_3(\frac{a_3}{a_1})(\frac{r_3}{r_1})^{n-1}}.$$

As $n \to \infty$,

$$(\frac{r_2}{r_1})^{n-1} = (\frac{r_3}{r_1})^{n-1} \to 0.$$

Hence,

$$lim_{n\to\infty}\frac{t_{n-1}}{t_n} = \frac{1}{r_1}.$$

(b)

$$\frac{t_n}{\sum_{k=1}^{n} t_k} = \frac{a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n}{a_1\sum_{k=1}^{n} {r_1}^k + a_2\sum_{k=1}^{n} {r_2}^k + a_3\sum_{k=1}^{n} {r_3}^k}.$$

This is simplified to

$$\frac{t_n}{\sum_{k=1}^{n} t_k} = \frac{1 + (\frac{a_2}{a_1})(\frac{r_2}{r_1})^n + (\frac{a_3}{a_1})(\frac{r_3}{r_1})^n}{\frac{r_1}{1-r_1}(\frac{1-{r_1}^n}{{r_1}^n}) + (\frac{a_2}{a_1})\frac{r_2}{1-r_2}(\frac{1-{r_2}^n}{{r_1}^n}) + (\frac{a_3}{a_1})\frac{r_3}{1-r_3}(\frac{1-{r_3}^n}{{r_1}^n})}.$$

Since

$$(\frac{r_2}{r_1})^{n-1} = (\frac{r_3}{r_1})^{n-1} \to 0 \quad \text{as } n \to \infty,$$

$$lim_{n\to\infty}\frac{t_n}{\sum_{k=1}^{n} t_k} = 1 - \frac{1}{r_1}.$$

Therefore, the limit exists.

$\square$

THEOREM 4.1.4.

(a) *For the Tribonacci sequence, let*

$$\tau = lim_{n\to\infty}\frac{t_{n-1}}{t_n}.$$

Then τ is the reciprocal of the Tribonacci constant.

(b) *For the infinite state version of the transition matrix in Theorem 4.1.1, the limiting vector is*

$$\nu = (1-\tau, (1-\tau)\tau, (1-\tau)\tau^2, (1-\tau)\tau^3, \dots).$$

PROOF.

(a) From the recursive relation of Tribonacci numbers,

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}.$$

$$1 = \tfrac{t_{n-1}}{t_n} + \tfrac{t_{n-2}}{t_n} + \tfrac{t_{n-3}}{t_n}.$$

Taking limits gives

$$1 = \tau + \tau^2 + \tau^3.$$

Solving for τ gives $\tau = 0.5437$ as the only positive real root of this equation and τ is the inverse of the Tribonacci constant τ^*.

(b) For the infinite state space, let ν be the limiting probability vector. Then,

$$\nu = (lim_{n\to\infty}\tfrac{t_n}{\sum_{k=1}^n t_k}, lim_{n\to\infty}\tfrac{t_{n-1}}{\sum_{k=1}^n t_k}, lim_{n\to\infty}\tfrac{t_{n-2}}{\sum_{k=1}^n t_k}, lim_{n\to\infty}\tfrac{t_{n-3}}{\sum_{k=1}^n t_k}, \dots).$$

Denote $\delta = lim_{n\to\infty}\frac{t_n}{\sum_{k=1}^n t_k}$. Then,

$$\nu = (\delta, \delta\tau, \delta\tau^2, \delta\tau^3, \dots).$$

But the probabilities must sum to 1, so

$$\delta + \delta\tau + \delta\tau^2 + \delta\tau^3 +, \dots = 1,$$

$$\delta = 1 - \tau.$$

The result follows. □

For the infinite state space, an example of this type of system is the $M/M^{[3]}/1$ queueing system where customers arrive singly but are served in batches of three, except if there are 1, 2, 0r 3 customers in the system. For $\lambda = \mu$, the probability of finding an empty system is 0.4563 (see Gross and Harris [**11**]).

4.1.2. Tetranacci Sequence.

A Tetranacci sequence is the sequence of integers formed by adding the preceding four consecutive terms.The first four numbers of the sequence are

$$T_0 = 0, T_1 = 1, T_2 = 1, T_3 = 2.$$

The sequence is governed by the recursive relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}; \quad n = 4, 5, \dots .$$

The first few terms of the Tetranacci sequence are

$$0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, \dots .$$

We present results about the class of transition matrices whose limiting probability vector is normalized sequence of Tetranacci numbers.

THEOREM 4.1.5.

For $0 < b \leq \frac{1}{2}$, consider the class of $n \times n$ transition matrices which have the same pattern as the 9×9 matrix below.

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 1-2b & b \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 1-b \end{pmatrix}.$$

The limiting probability vector ν for the $n \times n$ matrix is the normalized sequence of Tetranacci numbers

$$\nu = \frac{1}{\sum_{k=1}^{n} T_k}(T_n, T_{n-1}, T_{n-2}, \dots .T_2, T_1).$$

PROOF.

For the above matrix, let

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-2}, \nu_{n-1}).$$

Solving $\nu = \nu P$ gives

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+4}, \qquad k = 1, \ldots, n-5.$$

That is,

$$2\nu_k = \nu_{k-1} + \nu_{k+4}. \tag{12}$$

Using the reversal approach as shown in Theorem 3.0.2, we must show that

$$2T_k = T_{k+1} - T_{k-4}. \tag{13}$$

From the definition of Tetranacci numbers,

$$T_k = T_{k-1} + T_{k-2} + T_{k-3} + T_{k-4}$$
$$T_{k-1} = T_{k-2} + T_{k-3} + T_{k-4} + T_{k-5}.$$

Summing gives

$$T_k = 2T_{k-2} + 2T_{k-3} + 2T_{k-4} + T_{k-5}.$$

But $T_{k-1} - T_{k-5} = T_{k-2} + T_{k-3} + T_{k-4}$, so,

$$T_n = 2T_{n-1} - T_{n-5}.$$

This matches equation (13). The initial values of the sequence are accounted for in the last three columns. Hence the elements of the limiting probability vector are normalized Tetranacci numbers. □

The transition matrix is similar to the Tribonacci case except that there is yet another subdiagonal of almost all zero entries.

DEFINITION 4.1.6.

The Tetranacci constant $\kappa^* = lim_{n\to\infty}\frac{T_{n+1}}{T_n}$ *is the unique positive real root of*

$$x^4 - x^3 - x^2 - x - 1 = 0.$$

That is, $\kappa^* = 1.92756...$

THEOREM 4.1.7.

For Tetranacci sequence,

(a) $lim_{n\to\infty}\frac{T_{n-1}}{T_n}$ *exists,*

(b) $lim_{n\to\infty}\frac{T_n}{\sum_{k=1}^{n} T_k}$ *exists.*

PROOF.

For Tetranacci numbers,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}.$$

The resulting characteristic equation is given as

$$r^4 - r^3 - r^2 - r - 1 = 0.$$

Let the roots of this equation be r_1, r_2, r_3, r_4. By Descartes' Rule of Signs [**18**], there exists only one unique positive real root of the equation. Assume the roots are distinct. Then

$$T_n = a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n a_4{r_4}^n.$$

Let r_1 be the largest absolute value of the roots of the equation.

$$\frac{T_{n-1}}{T_n} = \frac{a_1{r_1}^{n-1} + a_2{r_2}^{n-1} + a_3{r_3}^{n-1} + a_4{r_4}^{n-1}}{a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n + a_4{r_4}^n}.$$

(a) Therefore,

$$\frac{T_{n-1}}{T_n} = \frac{1 + (\frac{a_2}{a_1})(\frac{r_2}{r_1})^{n-1} + (\frac{a_3}{a_1})(\frac{r_3}{r_1})^{n-1} + (\frac{a_4}{a_1})(\frac{r_4}{r_1})^{n-1}}{r_1 + r_2(\frac{a_2}{a_1})(\frac{r_2}{r_1})^{n-1} + r_3(\frac{a_3}{a_1})(\frac{r_3}{r_1})^{n-1} + r_4(\frac{a_4}{a_1})(\frac{r_4}{r_1})^{n-1}}.$$

As $n \to \infty$,

$$(\frac{r_2}{r_1})^{n-1} = (\frac{r_3}{r_1})^{n-1} = (\frac{r_4}{r_1})^{n-1} \to 0.$$

Therefore,

$$lim_{n\to\infty}\frac{T_{n-1}}{T_n} = \frac{1}{r_1}.$$

(b)

$$\frac{T_n}{\sum_{k=1}^{n} T_k} = \frac{a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n + a_4{r_4}^n}{a_1\sum_{k=1}^{n} {r_1}^k + a_2\sum_{k=1}^{n} {r_2}^k + a_3\sum_{k=1}^{n} {r_3}^k + a_4\sum_{k=1}^{n} {r_4}^k}.$$

This is simplified to

$$\frac{T_n}{\sum_{k=1}^{n} T_k} = \frac{1 + (\frac{a_2}{a_1})(\frac{r_2}{r_1})^n + (\frac{a_3}{a_1})(\frac{r_3}{r_1})^n + (\frac{a_4}{a_1})(\frac{r_4}{r_1})^n}{\frac{r_1}{1-r_1}(\frac{1-{r_1}^n}{{r_1}^n}) + (\frac{a_2}{a_1})\frac{r_2}{1-r_2}(\frac{1-{r_2}^n}{{r_1}^n}) + (\frac{a_3}{a_1})\frac{r_3}{1-r_3}(\frac{1-{r_3}^n}{{r_1}^n}) + (\frac{a_4}{a_1})\frac{r_4}{1-r_4}(\frac{1-{r_4}^n}{{r_1}^n})}.$$

As $n \to \infty$,

$$(\frac{r_2}{r_1})^{n-1} = (\frac{r_3}{r_1})^{n-1} = (\frac{r_3}{r_1})^{n-1} \to 0.$$

Therefore,

$$lim_{n\to\infty}\frac{T_n}{\sum_{k=1}^{n} T_k} = 1 - \frac{1}{r_1}.$$

$\square$

THEOREM 4.1.8.

(a) *For the class of transition matrices whose limiting probability vector forms the sequence of Tetranacci numbers, let*

$$\kappa = lim_{n\to\infty}\frac{T_{n-1}}{T_n}.$$

Then κ is the reciprocal of the Tetranacci constant.

(b) *For the infinite state version of the transition matrix in Theorem 4.1.5, the limiting vector is*

$$\nu = (1-\kappa, (1-\kappa)\kappa, (1-\kappa)\kappa^2, (1-\kappa)\kappa^3, ...).$$

PROOF.

(a) From the recursive relation of Tetranacci numbers,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}.$$

$$1 = \frac{T_{n-1}}{T_n} + \frac{T_{n-2}}{T_n} + \frac{T_{n-3}}{T_n} + \frac{T_{n-4}}{T_n}.$$

Taking limits gives

$$1 = \kappa + \kappa^2 + \kappa^3 + \kappa^4.$$

Solving for κ gives $\kappa = 0.5188$ and κ is the inverse of the Tetranacci constant κ^*.

(b) For the infinite state space,

$$\nu = (lim_{n\to\infty}\frac{T_n}{\sum_{k=1}^{n} T_k}, lim_{n\to\infty}\frac{T_n}{\sum_{k=1}^{n} T_k}, lim_{n\to\infty}\frac{T_n}{\sum_{k=1}^{n} T_k}, ...).$$

Denote $\alpha = lim_{n\to\infty}\frac{T_n}{\Sigma}$. Then,

$$\nu = (\alpha, \alpha\kappa, \alpha\kappa^2, \alpha\kappa^3, \alpha\kappa^4, ...).$$

But the probabilities must sum to 1, so

$$\alpha + \alpha\kappa + \alpha\kappa^2 + \alpha\kappa^3 +, ... = 1.$$

$$\alpha = 1 - \kappa.$$

The result follows. □

For the infinite state space, an example of this kind of system is the $M/M^{[4]}/1$ queuing system where the customers arrive one by one but they are served in batches of fours, except if there are 1, 2, or 3 customers in the system. For $\lambda = \mu$, the probability of having an empty system is approximately 0.4812.

4.1.3. Pentanacci Sequence.

A Pentanacci sequence is a generalization of the 5-step Fibonacci numbers with the 5 initial numbers given by

$$P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4.$$

The governing recursive relation is given as

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}; \quad n = 5, 6 \ldots .$$

The first few numbers of the Pentanacci sequence are:

1, 1, 2, 4, 8, 16, 31, 61, 120, 236,... .

THEOREM 4.1.9.

For $0 < b \leq \frac{1}{2}$, consider the class of $n \times n$ transition matrices which have the same pattern as the 9×9 matrix below.

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 1-2b & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 1-b \end{pmatrix}$$

The limiting probability vector ν for the $n \times n$ matrix is the normalized sequence of Pentanacci numbers

$$\nu = \frac{1}{\sum_{k=1}^{n} P_k}(P_n, P_{n-1}, P_{n-2}, ...P_2, P_1).$$

PROOF.

For the above matrix, let

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-2}, \nu_{n-1}).$$

Solving $\nu = \nu P$ gives

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+5}, \qquad k = 1, \ldots, n-6.$$

That is,

$$2\nu_k = \nu_{k-1} + \nu_{k+5}.$$

Using the reversal as shown in the proof of Theorem 3.0.2, we must show that

$$2P_k = P_{k+1} + P_{k-5} \tag{14}$$

From the definition of Pentanacci numbers,

$$P_k = P_{k-1} + P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5}.$$
$$P_{k-1} = P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5} + P_{k-6}.$$

So,

$$P_k = 2P_{k-2} + 2P_{k-3} + 2P_{k-4} + 2P_{k-5} + P_{k-6},$$

But $P_{k-1} - P_{k-6} = P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5}$, therefore,

$$P_k = 2P_{k-1} - P_{k-6}$$

This matches equation (14). The initial values of sequence are contained in the last four columns. Hence the elements of the limiting probability vector are normalized Pentanacci numbers. □

DEFINITION 4.1.10.

The Pentanacci constant $\lambda^* =$ *is the unique positive real root of*

$$x^5 - x^4 - x^3 - x^2 - x - 1 = 0.$$

That is, $\lambda^* = 1.96594823...$

THEOREM 4.1.11.

For Pentanacci sequence,

(a) $lim_{n\to\infty}\frac{P_{n-1}}{P_n}$ *exists,*

(b) $lim_{n\to\infty}\frac{P_n}{\sum_{k=1}^{n}P_k}$ *exists.*

PROOF.

For Pentanacci numbers,

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}.$$

The resulting characteristic equation is given as

$$r^5 - r^4 - r^3 - r^2 - r - 1 = 0.$$

Let the roots of this equation be $r_1, r_2, ..., r_5$. By Descartes' Rule of Signs [**18**], there exists only one unique positive real root of the equation. Assume the roots are distinct

$$P_n = a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n + a_4{r_4}^n + a_5{r_5}^n.$$

Let r_1 be the largest absolute value of the roots of the equation.

$$\frac{P_{n-1}}{P_n} = \frac{a_1{r_1}^{n-1} + a_2{r_2}^{n-1} + a_3{r_3}^{n-1} + a_4{r_4}^{n-1} + a_5{r_5}^{n-1}}{a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n + a_4{r_4}^n + a_5{r_5}^n}.$$

(a) Therefore,

$$\frac{P_{n-1}}{P_n} = \frac{1 + (\frac{a_2}{a_1})(\frac{r_2}{r_1})^{n-1} + (\frac{a_3}{a_1})(\frac{r_3}{r_1})^{n-1} + (\frac{a_4}{a_1})(\frac{r_4}{r_1})^{n-1} + (\frac{a_5}{a_1})(\frac{r_5}{r_1})^{n-1}}{r_1 + r_2(\frac{a_2}{a_1})(\frac{r_2}{r_1})^{n-1} + r_3(\frac{a_3}{a_1})(\frac{r_3}{r_1})^{n-1} + r_4(\frac{a_4}{a_1})(\frac{r_4}{r_1})^{n-1} + r_5(\frac{a_5}{a_1})(\frac{r_5}{r_1})^{n-1}}.$$

As $n \longrightarrow \infty$,

$$(\frac{r_j}{r_1})^{n-1} \longrightarrow 0 \quad j = 2, ..., 5.$$

Therefore,

$$lim_{n\to\infty}\frac{P_{n-1}}{P_n} = \frac{1}{r_1}.$$

(b)

$$\frac{P_n}{\sum_{k=1}^{n} P_k} = \frac{a_1{r_1}^n + a_2{r_2}^n + a_3{r_3}^n + a_4{r_4}^n + a_5{r_5}^n}{a_1\sum_{k=1}^{n} {r_1}^k + a_2\sum_{k=1}^{n} {r_2}^k + a_3\sum_{k=1}^{n} {r_3}^k + a_4\sum_{k=1}^{n} {r_4}^k + a_5\sum_{k=1}^{n} {r_5}^k}.$$

This is simplified to

$$\frac{P_n}{\sum_{k=1}^{n} P_k} = \frac{1 + (\frac{a_2}{a_1})(\frac{r_2}{r_1})^n + (\frac{a_3}{a_1})(\frac{r_3}{r_1})^n + (\frac{a_4}{a_1})(\frac{r_4}{r_1})^n}{\frac{r_1}{1-r_1}(\frac{1-{r_1}^n}{{r_1}^n}) + (\frac{a_2}{a_1})\frac{r_2}{1-r_2}(\frac{1-{r_2}^n}{{r_1}^n}) + (\frac{a_3}{a_1})\frac{r_3}{1-r_3}(\frac{1-{r_3}^n}{{r_1}^n}) + (\frac{a_4}{a_1})\frac{r_4}{1-r_4}(\frac{1-{r_4}^n}{{r_1}^n}) + (\frac{a_5}{a_1})\frac{r_5}{1-r_5}(\frac{1-{r_5}^n}{{r_1}^n})}.$$

As $n \to \infty$

$$(\frac{r_j}{r_1})^{n-1} \to 0. \quad j = 2, ..., 5.$$

Therefore,

$$lim_{n\to\infty}\frac{P_n}{\sum_{k=1}^{n} P_k} = 1 - \frac{1}{r_1}.$$

□

THEOREM 4.1.12.

(a) *For the class of transition matrices whose limiting probability vector forms the sequence of Pentanacci numbers, let*

$$\lambda = lim_{n\to\infty}\frac{P_{n-1}}{P_n}.$$

Then, λ is the reciprocal of the Pentanacci constant.

(b) *For the infinite state version of the transition matrix in Theorem 4.1.9, the limiting vector is*

$$\nu = (1-\lambda, (1-\lambda)\lambda, (1-\lambda)\lambda^2, (1-\lambda)\lambda^3, ...).$$

PROOF.

(a) From the recursive relation of Pentanacci numbers,

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}.$$

$$1 = \frac{P_{n-1}}{P_n} + \frac{P_{n-2}}{P_n} + \frac{P_{n-3}}{P_n} + \frac{P_{n-4}}{T_n} + \frac{P_{n-5}}{T_n}.$$

Taking limits gives

$$1 = \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5$$

Solving for κ gives $\lambda = 0.5087$ and λ is the inverse of the Pentanacci constant λ^*.

(b) For the infinite state space, let ν be the limiting probability vector, then

$$\nu = (lim_{n\to\infty}\frac{P_n}{\sum_{k=1}^{n} P_k}, lim_{n\to\infty}\frac{P_{n-1}}{\sum_{k=1}^{n} P_k}, lim_{n\to\infty}\frac{P_{n-2}}{\sum_{k=1}^{n} P_k}, \ldots).$$

Denote $\beta = lim_{n\to\infty}\frac{P_n}{\sum_{k=1}^{n} P_k}$. Then,

$$\nu = (\beta, \beta\lambda, \beta\lambda^2, \beta\lambda^3, \beta\lambda^4, \ldots).$$

But the probabilities must sum to 1, so

$$\beta + \beta\lambda + \beta\lambda^2 + \beta\lambda^3 +, \ldots = 1,$$

$$\beta = 1 - \lambda.$$

The result follows. □

For the infinite state space, an example of this type of system is the $M/M^{[5]}/1$ queuing system where the customers arrive one by one but they are served in batches of fives. except if there are 1, 2, 3, or 4 customers in the system. For $\lambda = \mu$, the probability of having an empty system is approximately 0.4913.

We can generalize the Tribonacci, Tetranacci, Pentanacci numbers to r-step Fibonacci numbers $\{x_n\}$, $n = 0, 1, 2, \ldots$.

THEOREM 4.1.13.

For $b \leq \frac{1}{2}$, we consider the class of infinite state space probability transition matrices whose finite state space probability transition matrix has a limiting probability vector forming the sequence of normalized r-step Fibonacci numbers. As $r \to \infty$, the limiting probability vector of the infinite state space system becomes

$$\nu = (0.5, 0.5^2, 0.5^3, 0.5^4, \ldots).$$

PROOF.

For the r-step Fibonacci sequence the recursive relation is given as

$$x_n = x_{n-1} + x_{n-2} + x_{n-3} + ... + x_{n-r}.$$

$$1 = \frac{x_{n-1}}{x_n} + \frac{x_{n-2}}{x_n} + \frac{x_{n-3}}{x_n} + \frac{x_{n-4}}{x_n} + \cdots + \frac{x_{n-r}}{x_n}.$$

We assume $lim_{n\to\infty}\frac{x_{n-1}}{x_n} = \theta$ exists. Taking the limit as $n \to \infty$ gives

$$1 = \theta + \theta^2 + \theta^3 + \theta^4 + ... + \theta^r.$$

As $r \to \infty$,

$$1 = \theta(1 + \theta^2 + \theta^3 + \theta^4 + ...).$$

This implies,

$$1 = \frac{\theta}{1-\theta},$$

$$\theta = 0.5,$$

so

$$\nu = (1-\theta, (1-\theta)\theta, (1-\theta)\theta^2, (1-\theta)\theta^3, (1-\theta)\theta^4, ...),$$

and the result is proved. □

For the finite state space, the class of transition matrices whose limiting probability vector is the normalized r-step Fibonacci sequence, is an example of $M/M^{[r]}/1$ queueing systems.

CHAPTER 5

Limiting Vector of the Generalized Fibonacci Sequences

Consider the sequence formed when the initial conditions of the Fibonacci sequences are generalized, that is, $F_0 = x$, $F_1 = y$. The class of sequence formed is also known as the Generalized Fibonacci sequence(or G-sequence). Mathematically ,

$$G_n = G_{n-1} + G_{n-2}; \quad n = 2, 3, \ldots .$$

where $G_0 = x$ and $G_1 = y$. A.F Horadam [**12**] showed that the generalized Fibonacci sequences are related to Fibonacci Sequences. His result is stated below.

PROPERTY 5.0.14.

For a generalized Fibonacci sequence $G_{x,y,k} \equiv G_k$ with the initial conditions $G_0 = x$, $G_1 = y$,

$$G_k = xF_{k-1} + yF_k, \qquad k = 1, 2, \ldots .$$

In this section we investigate the class of transition matrices whose limiting probability vector is the normalized sequence of every (every other) number that belongs to the generalized Fibonacci numbers. We also investigate some properties of these classes of transition matrices.

THEOREM 5.0.15.

For $0 < b \leq min\{\frac{1}{3}, \frac{y}{x+2y}\}$, the class of $n \times n$ transition matrices whose limiting probability vector is the normalized sequence of "every other" (odd indexed) number of the sequence $\{G_{x,y,i}\}$ belonging to the generalized Fibonacci sequence, is given as

$$\begin{pmatrix} 1-b & b & 0 & . & . & . & . & 0 \\ 2b & 1-3b & b & . & . & . & . & 0 \\ b & b & 1-3b & b & . & . & . & 0 \\ b & 0 & b & 1-3b & . & . & . & 0 \\ b & 0 & 0 & b & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ b & . & . & . & . & b & 1-3b & b \\ 3b & 0 & . & . & . & . & b & 1-(\frac{x+2y}{y})b \end{pmatrix}.$$

Then,

$$\nu = \frac{1}{\sum_{i=1}^{n} G_{2i-1}}(G_{2n-1}, G_{2n-3}, ..., G_3, G_1).$$

(b) On the other hand, for $0 < b \leq \frac{1}{3}$, the class of $n \times n$ transition matrices given below has the limiting probability vector formed by the even indexed numbers of the sequence $\{G_{x,y,i}\}$.

$$P = \begin{pmatrix} 1-b & b & 0 & 0 & . & . & . & . & 0 \\ 2b & 1-3b & b & . & . & . & . & . & 0 \\ b & b & 1-3b & b & . & . & . & . & 0 \\ b & 0 & b & 1-3b & b & . & . & . & 0 \\ b & 0 & 0 & b & 1-3b & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ b & . & . & . & . & . & b & 1-3b & b \\ 3b & 0 & . & . & . & . & 0 & b & 1-(\frac{2x+3y}{x+y})b \end{pmatrix}.$$

That is,

$$\nu = \frac{1}{\sum_{i=1}^{n} G_{2i}}(G_{2n}, G_{2n-2}, ..., G_4, G_2).$$

PROOF.

For the generalized Fibonacci sequences, the first few terms are

$$x, y, (x+y), (x+2y), 2x+3y, 3x+5y, \dots .$$

Let the limiting probability vector be given as

$$\nu = (\nu_0, \nu_1, \nu_2, \dots, \nu_{n-1}).$$

We solve $\nu = \nu P$. The balance equation for state $k = 1, 2, \dots, n-2$ is

$$\nu_k = b\nu_{k-1} + (1-3b)\nu_k + b\nu_{k+1}.$$

This implies

$$\nu_{k+1} = 3\nu_k - \nu_{k-1}. \tag{15}$$

Using reversal and increasing the index spread as before, we must show that

$$G_{k-2} = 3G_k - G_{k+2}. \tag{16}$$

From the definition of the generalized Fibonacci numbers,

$$\begin{aligned} G_k &= G_{k-1} + G_{k-2}, \\ G_{k-1} &= G_{k-2} + G_{k-3}, \\ G_{k-3} &= G_{k-2} - G_{k-4}, \end{aligned}$$

Summing these, we have

$$G_k = 3G_{k-2} - G_{k-4}.$$

This matches equation (16), and the last columns for the two transition matrices are formed by substituting the ratio of the fifth(fourth) to the third(second) terms of the sequence respectively. These give the initial values of the sequence. Hence the two transition matrices produce normalized sequences of odd indexed and even indexed generalized Fibonacci numbers respectively. □

THEOREM 5.0.16.

For $0 < b \leq min\{\frac{1}{2}, \frac{y}{x+y}\}$, *the class of* $n \times n$ *transition matrices, whose limiting probability vector is formed by every number in the sequence* $\{G_{x,y,i}\}$ *belonging to the generalized Fibonacci series, is given as*

$$\begin{pmatrix} 1-b & b & 0 & . & 0 & 0 & 0 & 0 \\ b & 1-2b & b & . & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & . & b & 0 & 0 & 0 \\ 0 & b & 0 & . & 1-2b & b & 0 & 0 \\ . & . & . & . & . & .. & . & . \\ . & . & . & . & . & .. & . & . \\ . & . & . & . & b & 0 & 1-2b & b \\ 0 & . & . & . & 0 & b & \frac{x}{y}b & 1-(\frac{x+y}{y})b \end{pmatrix}.$$

PROOF.

Let the limiting probability vector be

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

We solve $\nu = \nu P$. The balance equation for state k, $k = 1, 2, \ldots, n-3$ is

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+2}.$$

So

$$2\nu_k = \nu_{k-1} + \nu_{k+2}. \tag{17}$$

Using reversal as described earlier in Theorem 3.0.2,we need to show that

$$2G_k = G_{k+1} + G_{k-2}. \tag{18}$$

Now, from the definition of the generalized Fibonacci numbers, we have

$$G_k = G_{k-1} + G_{k-2},$$
$$G_{k-2} = G_{k-1} - G_{k-3}.$$

Sum these two equations to get

$$G_k = 2G_{k-1} - G_{k-3}.$$

This corresponds to (18), and the initial condition is given by the last column, and also the last row. Hence the limiting probability vector forms the normalized sequence of consecutive generalized Fibonacci numbers. $\square$

To illustrate the above results, we consider the Lucas sequence as an example in the next section.

EXAMPLE 5.0.1. *(Lucas Sequence)*

Consider the sequence formed when $x = 2$, *and* $y = 1$ *and governed with the recursive relation*

$$L_n = L_{n-1} + L_{n-2}. \quad n = 2, 3, \dots .$$

This sequence is known as the Lucas Sequence[12]. The first few terms of the sequence are:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \dots .$$

For $x = 2$, $y = 1$, *and* $0 < b \leq \frac{1}{3}$, *the class of* 7×7 *transition matrices of the form*

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & b & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & b & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & b & 0 & 1-2b & b \\ 0 & 0 & 0 & 0 & b & 2b & 1-3b \end{pmatrix}$$

has the limiting probability vector given as

$$\nu = \tfrac{1}{73}(29, 18, 11, 7, 4, 3, 1).$$

That is, the limiting probability vector forms the sequence of consecutive Lucas Numbers.

On the other hand, for $0 < b \leq \frac{1}{4}$*, the class of transition matrices whose limiting probability vector forms the sequence of every other Lucas number is given by*

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 \\ 2b & 1-3b & b & 0 & 0 & 0 & 0 \\ b & b & 1-3b & b & 0 & 0 & 0 \\ b & 0 & b & 1-3b & b & 0 & 0 \\ b & 0 & 0 & b & 1-3b & b & 0 \\ b & 0 & 0 & 0 & b & 1-3b & b \\ 3b & 0 & 0 & 0 & 0 & b & 1-4b \end{pmatrix}.$$

That is,

$$\nu = \tfrac{1}{841}(521, 199, 76, 29, 11, 4, 1).$$

THEOREM 5.0.17.

For the generalized Fibonacci number sequence,

$$lim_{k\to\infty} \tfrac{G_{2n-1}}{\sum_{k=1}^{n} G_{2k-1}}$$

exists.

PROOF.

For generalized Fibonacci numbers,

$$G_k = G_{k-1} + G_{k-2} = xF_{k-1} + yF_k.$$

Also,

$$G_{2k-1} = G_{2k-2} + G_{2k-1} = xF_{2k-2} + yF_{2k-1}.$$

$$\frac{G_{2n-1}}{\sum_{k=1}^{n} G_{2k-1}} = \frac{xF_{2n-2} + yF_{2n-1}}{x\sum_{k=1}^{k=n} F_{2k-2} + y\sum_{k=1}^{n} G_{2k-1}}.$$

But $\sum_{k=1}^{k=n} F_{2k-2} = F_{2n-1} - 1$, and $\sum_{k=1}^{k=n} F_{2k-1} = F_{2n}$. Therefore,

$$\frac{G_{2n-1}}{\sum_{k=1}^{n} G_{2k-1}} = \frac{xF_{2n-2} + yF_{2n-1}}{x(F_{2n-1} - 1) + yF_{2n}}.$$

Taking the limits,

$$lim_{k\to\infty} \frac{G_{2n-1}}{\sum_{k=1}^{n} G_{2k-1}} = \frac{x\ell^2 + y\ell}{x\ell + y} = \ell.$$

Therefore the limit exists. □

THEOREM 5.0.18. *(Infinite State Space)*

Assume we have sequence of integers that belongs to the generalized Fibonacci class of sequences with the initial values $G_0 = i$, and $G_1 = j$. For the infinite state space,

(a) *The limiting probability vector of the class of infinite state space transition matrices in theorem (5.0.16) becomes*

$$\nu = (\ell^2, \ell^3, \ell^4, \ell^4, ...)$$

where $\ell = \frac{\sqrt{5}-1}{2}$.

(b) *The limiting probability vector of the class of infinite state space transition matrices in Theorem 5.0.15 is*

$$\nu = (\ell, \ell^3, \ell^5, \ell^7, ...).$$

PROOF.

(a) For initial values i, j, the resulting sequence is given by

$$i, j, (i+j), (i+2j), (2i+3j), \ldots .$$

From the definition of the generalized Fibonacci numbers,

$$G_k = G_{k-1} + G_{k-2}.$$

We first show that the $lim_{k\to\infty}\frac{G_{k-1}}{G_k}$ exists. Let $L^* = lim_{k\to\infty}\frac{G_{k-1}}{G_k}$. Then, Property 5.0.14, implies

$$L^* = lim_{k\to\infty}[\tfrac{iF_{k-2}+jF_{k-1}}{iF_{k-1}+jF_k}].$$

Since $\ell = lim_{k\to\infty}\{\frac{F_{k-1}}{F_k}\}$, then

$$L^* = \tfrac{i\ell^2+j\ell}{i\ell+j} = \ell.$$

The limiting probability vector of the finite state version,

$$\nu = \tfrac{1}{\sum_{k=1}^{k=n} G_k}(G_n, G_{n-1}, G_{n-2}, \ldots, G_2, G_1).$$

For the infinite state space, let $\Sigma = \sum_{k=1}^{n} G_k$ and let $\kappa = lim_{n\to\infty}\frac{G_n}{\Sigma}$. Then,

$$\nu = (lim_{n\to\infty}\tfrac{G_n}{\sum_{k=1}^{k=n} G_k}, lim_{n\to\infty}\tfrac{G_{n-1}}{\sum_{k=1}^{k=n} G_k}, lim_{n\to\infty}\tfrac{G_{n-2}}{\sum_{k=1}^{k=n} G_k}, \ldots).$$

Therefore, ν forms a geometric sequence given by

$$\nu = (\kappa, \kappa\ell, \kappa\ell^2, \ldots).$$

Since $\kappa(1 + \ell + \ell^2 + \ell^3 + \ldots) = 1$, then

$$\kappa = 1 - \ell = \ell^2.$$

The resulting limiting probability vector is given as

$$\nu = (\ell^2, \ell^3, \ell^4, \ldots).$$

(b) Similarly, for the class of finite state space transition matrices whose limiting probability vector is formed by the sequence of every other number belonging to the class of generalized Fibonacci sequence,

$$\nu = \tfrac{1}{\sum_{k=1}^{k=n} G_{2k-1}}(G_{2n-1}, G_{2n-3}, G_{2n-5}, \ldots, G_3, G_1).$$

For the infinite state space, let $\Sigma^* = \sum_{k=1}^{n} G_{2k-1}$, and $\omega_1 = lim_{n\to\infty}\frac{G_{2n-1}}{\Sigma^*}$. Then

$$\nu = (\omega_1, \omega_1\ell^2, \omega_1\ell^4, \ldots).$$

Hence,

$$\omega_1 = 1 - \ell^2 = \ell,$$

and the result is proved. □

This implies that for the infinite state space and for different choices of initial values G_0 and G_1 , the resulting limiting probability vectors for the two classes of matrices are the same as those in Theorems 3.0.7 and 4.0.11.

5.1. The n-step Generalized Fibonacci Sequence

This is a broad class of sequences of integers formed by adding the n-preceding terms given n starting values. We illustrate this class by considering the classes of transition matrices whose limiting probability vectors form normalized sequences of the generalized Tribonacci numbers and the generalized Tetranacci numbers.

5.1.1. The generalized Tribonacci Sequences.

This is a class of sequences of numbers formed when the preceding three terms are added together. Given the starting values $x > 0$, $y > 0$, $z > 0$, the 3-step generalized Fibonacci sequences are governed by the recursive relation:

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}; \quad n = 3, 4, \dots .$$

The first few terms of the sequence are $t_0, t_1, t_2, \dots$. That is,

$$x, y, z, (x+y+z), (x+2y+2z), (2x+3y+4z), (4x+6y+7y), (7x+11y+13y), \dots .$$

THEOREM 5.1.1.

The class of $n \times n$ transition matrices, which has the same pattern as the 8×8 transition matrices given below, has the limiting probability vector formed by the normalized sequence of the first n generalized Tribonacci numbers; $t_n, \dots, t_1$.

(a) *When* $\frac{z}{y} \leq 1$, *and* $0 < b \leq min\{\frac{1}{2}, \frac{y}{z}, \frac{z}{x+y+z}\}$

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & b & \frac{y-z}{z}b & \frac{x}{z}b & 1-\frac{x+y+z}{z}b & b \\ 0 & 0 & 0 & 0 & \frac{z}{y}b & 0 & 0 & 1-\frac{z}{y}b \end{pmatrix},$$

(b) *when* $z \geq x+y$, *and* $0 < b \leq min\{\frac{1}{2}, \frac{y}{z}, \frac{z}{x+y+z}\}$

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & \frac{x+y}{z}b & 0 & 0 & 1-\frac{x+y+z}{z}b & b \\ 0 & 0 & 0 & \frac{z-x-y}{y}b & b & \frac{x}{y}b & 0 & 1-\frac{z}{y}b \end{pmatrix},$$

(c) *when* $y \leq z < x+y$, *and* $0 < b \leq min\{\frac{1}{2}, \frac{y}{z}, \frac{z}{x+y+z}\}$

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & \frac{y}{z}b & 0 & \frac{x}{z}b & 1-\frac{x+y+z}{z} & b \\ 0 & 0 & 0 & \frac{z-y}{y}b & b & 0 & 0 & 1-\frac{z}{y}b \end{pmatrix},$$

then

$$\nu = \frac{1}{\sum_{i=1}^{k=n} t_k}(t_n, t_{n-1}, t_{n-2}, ..., t_3, t_2, t_1).$$

PROOF.

Let the limiting probability vector ν be

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

We solve $\nu = \nu P$. The balance equation for state k, $k = 1, 2, \ldots, n-6$ is

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+3}.$$

So,

$$2\nu_k = \nu_{k-1} + \nu_{k+3}. \tag{19}$$

Using reversal, we must show that

$$2t_k = t_{k+1} + t_{k-3}. \tag{20}$$

Now, from the definition of the generalized Tribonacci numbers, we have

$$t_k = t_{k-1} + t_{k-2} + t_{k-3},$$
$$t_{k-1} = t_{k-2} + t_{k-3} + t_{k-4},$$

Sum these to get

$$t_k = 2t_{k-2} + 2t_{k-3} + t_{k-4}.$$

But $t_{k-1} - t_{k-4} = t_{k-2} + t_{k-3}$, so,

$$t_k = 2t_{k-1} - t_{k-4}.$$

This matches equation (20), and the initial conditions of the sequence are given by the last five columns. Hence the limiting probability vector forms the normalized sequence of consecutive generalized Tribonacci numbers. □

5.1.2. The generalized Tetranacci Sequence.

Similar to the 3-step generalized Fibonacci sequence is the 4-step generalized Fibonacci sequence. It is a class of sequence formed by adding the preceding four integers. Given the initial values $x > 0$, $y > 0$, $z > 0$, w , this class of sequence is governed by the recursive relation:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}. \quad n = 4, 5, \ldots .$$

We consider a special case when $w = x + y + z$. The first few terms of the sequence are:

$$x, y, z, (x+y+z), (2x+2y+2z), (3x+4y+4z), (6x+7y+8y), (12x+14y+15z), \ldots .$$

We investigate the class of transition matrices whose limiting probability vector is formed by the normalized sequence of thes4e generalized Tetranacci numbers.

THEOREM 5.1.2.

The class of $n \times n$ transition matrices, which has the same pattern as the 8×8 transition matrices given below, has the limiting probability vector formed by every number of the generalized Tetranacci sequence.

(a) *When* $\frac{z}{y} \leq 1$, *and* $0 < b \leq min\{\frac{1}{2}, \frac{y}{z}, \frac{z}{x+y+z}\}$

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & b & \frac{y-z}{z}b & \frac{x}{z}b & 0 & 1-\frac{x+y+z}{z}b & b \\ 0 & 0 & 0 & \frac{z}{y}b & 0 & 0 & 0 & 1-\frac{z}{y}b \end{pmatrix},$$

(b) *when* $z \geq x + y$, *and* $0 < b \leq min\{\frac{1}{2}, \frac{y}{z}, \frac{z}{x+y+z}\}$

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & \frac{x+y}{z}b & 0 & 0 & 0 & 1-\frac{x+y+z}{z}b & b \\ 0 & 0 & \frac{z-x-y}{y}b & b & \frac{x}{y}b & 0 & 0 & 1-\frac{z}{y}b \end{pmatrix},$$

(c) *when* $y \leq z < x+y$, *and* $0 < b \leq min\{\frac{1}{2}, \frac{y}{z}, \frac{z}{x+y+z}\}$

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & b & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1-2b & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 1-2b & b & 0 \\ 0 & 0 & \frac{y}{z}b & 0 & \frac{x}{z}b & 0 & 1-\frac{x+y+z}{z} & b \\ 0 & 0 & \frac{z-y}{y}b & b & 0 & 0 & 0 & 1-\frac{z}{y}b \end{pmatrix},$$

then

$$\nu = \frac{1}{\sum_{i=1}^{n} T_k}(T_n, T_{n-1}, T_{n-2}, ..., T_3, T_2, T_1).$$

PROOF.

Let the limiting probability vector ν be

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

We solve $\nu = \nu P$. The balance equation for state k, $k = 1, 2, \ldots, n-7$ is

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+4}. \qquad (21)$$

That is,

$$2\nu_k = \nu_{k-1} + \nu_{k+4}. \tag{22}$$

Using our reversal method, we must show that

$$2T_{k-1} = T_k + T_{k-4}. \tag{23}$$

From the definition of the generalized Tetranacci numbers, we have

$$T_k = T_{k-1} + T_{k-2} + T_{k-3} + T_{k-4},$$
$$T_{k-1} = T_{k-2} + T_{k-3} + T_{k-4} + T_{k-5},.$$

Sum these to get

$$T_k = 2T_{k-2} + 2T_{k-3} + 2T_{k-4} + T_{k-5}.$$

But $T_{k-1} - T_{k-5} = T_{k-2} + T_{k-3} + T_{k-4}$, so,

$$T_k = 2T_{k-1} - T_{k-5}.$$

This corresponds to (23), and the initial conditions are given by the last six columns. Hence the limiting probability vector forms the normalized sequence of consecutive generalized Tetranacci numbers. □

Also to be noted is the fact that for the infinite state space, the transition matrix becomes the model for the $M/M^{[4]}/1$ when $\mu = \lambda$. This result can be generalized to find the class of transition matrices whose limiting probability vector forms the normalized sequence of the n-step generalized Fibonacci numbers. Also to be be noted is the fact that for the infinite state space case of these classes of transition matrices, the limiting probability vector is the same as the limiting vector for the corresponding n-step Fibonacci numbers (for infinite state space).

5.2. On Partial Sums of Sequences

For example, consider a 5×5 transition matrix

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}.$$

The resulting limiting probability is

$$\nu = \frac{1}{26}(12, 7, 4, 2, 1).$$

The limiting probability vector is observed to form the normalized sequence of the cumulative sums of Fibonacci numbers. On the other hand, it will be observed that by changing the (5,5) entry of the matrix to $\frac{3}{4}$ and the the (5,1) entry to 0,the limiting probability vector forms the normalized sequence of Fibonacci numbers. Investigating this interesting property further, we obtain the following result.

THEOREM 5.2.1.

For $0 < b \leq \frac{1}{2}$, the $n \times n$ class of transition probability matrices with the same pattern as the 8×8 transition matrix

$$P = \begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 1-2b & b \\ b & 0 & 0 & 0 & 0 & b & 0 & 1-2b \end{pmatrix},$$

has the limiting probability vector of the normalized sequence of the cumulative sums of Fibonacci numbers. That is,

$$\nu = \frac{1}{\sum_{j=i}^{n}\sum_{i=1}^{j} F_i}\left(\sum_{i=0}^{n} F_i, \sum_{i=0}^{n-1} F_i, \sum_{i=0}^{n-2} F_i, \ldots \sum_{i=0}^{2} F_i, \sum_{i=0}^{1} F_i\right).$$

We delay the proof since we will prove a more general version in our next theorem.

Since $\sum_{i=0}^{j} F_i = F_{j+2} - 1$ and $\sum_{j=0}^{n}\sum_{i=1}^{j} F_i = \sum_{j=1}^{n}[F_{j+2} - 1] = F_{n+4} - (n+3)$, we have the following result.

COROLLARY 5.2.2.

The limiting probability vector can be written as

$$\nu = \frac{1}{F_{n+4}-(n+3)}(F_{n+2} - 1, F_{n+1} - 1, F_n - 1, \ldots, F_4 - 1, F_3 - 1)$$

Also for the generalized Fibonacci sequences we have the following result.

THEOREM 5.2.3.

For the class of generalized Fibonacci sequences, the class of $n \times n$ transition matrices of type illustrated by the 8×8 matrix

$$M = \begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1-2b & b & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1-2b & b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 1-2b & b & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 1-2b & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 1-2b & b & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 1-2b & b \\ \frac{y}{x+y}b & 0 & 0 & 0 & 0 & b & [\frac{x}{x+y}]b & 1-2b \end{pmatrix},$$

has the limiting probability vector of the normalized sequence of cumulative sums of the Fibonacci numbers. That is,

$$\nu = \frac{1}{\sum_{j=1}^{j=n} C_j}(C_n, C_{n-1}, C_{n-2}, \ldots, C_2, C_1)$$

where $C_j = \sum_{i=0}^{j} G_i$.

PROOF.

Given the generalized Fibonacci sequence $\{G_i\}$ with the first few terms

$$x, y, (x+y), (x+2y), (2x+3y), (3x+5y), ...,$$

the sequence of cumulative sums is given as

$$x, (x+y), 2(x+y), (3x+4y), (5x+7y), (8x+12y), \ldots .$$

Let the limiting probability vector be given as

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-2}, \nu_{n-1}).$$

Then, solving for ν gives the balance equation

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+2}.$$

That is, for $k = 1, 2, \ldots, n-3$,

$$\nu_{k+2} = 2\nu_k - \nu_{k-1}. \tag{24}$$

Reversing the order means that we must show

$$\sum_{i=0}^{k-1} G_i = 2\sum_{i=0}^{k+1} G_i - \sum_{i=0}^{k+2} G_i.$$

The right hand side is

$$2(\sum_{i=0}^{k-1} G_i) + 2G_k + 2G_{i+1} - \sum_{i=0}^{k-1} G_i - G_k - G_{k+1} - G_{k+2} = \sum_{i=0}^{k-1} G_i.$$

Simplifying these terms gives

$$G_{k+2} = G_{k+1} + G_k. \tag{25}$$

But (25) still holds by the definition of generalized Fibonacci numbers. For the last two columns of the transition matrix M, the coefficient of b in $(n-1, n-1)$ entry of the transition matrix is found to be the ratio of the third to the second term of the sequence. In a similar manner the $(n-1, n-2)$ entry is found by subtracting the fourth term of the sequence from 4. The $(n-1, 0)$ entry of the matrix is found using the probabilistic assumption that the rows of a transition matrix must sum up to 1. Therefore, the limiting probability vector of the class of transition matrices M forms the normalized sequence of the cumulative sums of generalized Fibonacci numbers. □

Examining this class of transition matrices M further, it is observed that for the infinite state space, the transition matrix is exactly the same as the infinite transition matrix for the Fibonacci numbers. Hence for the infinite state space, the limiting probability vector is equal to the one given in Theorem 5.0.18.

CHAPTER 6

Limiting Vector of some other Sequences

In this chapter, we consider some classes of transition matrices whose limiting probability vector forms the sequence of some well known number sequences. We present results about the class of transition matrices whose limiting probability vector is the normalized vector of these sequences. The infinite state version of these classes are also presented.

6.1. Jacobsthal Sequence

The Jacobsthal sequence [**14**], named after the German-born mathematician Ernst Jacobsthal (1882-1965), is a sequence of integers whose initial values are $J_0 = 0$ and $J_1 = 1$ and governed by the recursive relation

$$J_n = J_{n-1} + 2J_{n-2}; \quad n = 2, 3, \ldots . \tag{26}$$

The first few few terms of the sequence are:

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \ldots .$$

We present results concerning the class of transition matrices whose limiting probability vector forms the normalized sequence of Jacobsthal numbers.

THEOREM 6.1.1.

For $0 < b \leq \frac{1}{5}$, we consider the class of $n \times n$ transition matrices of the form

$$\begin{pmatrix} 1-b & b & 0 & 0 & . & . & 0 & 0 & 0 & \\ 4b & 1-5b & b & 0 & . & . & 0 & 0 & . & 0 \\ 0 & 4b & 1-5b & b & . & . & 0 & 0 & 0 & \\ 0 & 0 & 4b & 1-5b & . & . & 0 & 0 & 0 & \\ . & . & . & . & . & . & .. & . & . & \\ . & . & . & . & . & . & .. & . & . & \\ 0 & 0 & . & . & . & . & 1-5b & b & 0 & \\ 0 & . & . & . & . & 0 & 4b & 1-5b & b & \\ b & 0 & . & . & . & 0 & 0 & 4b & 1-5b & \end{pmatrix}.$$

The corresponding limiting probability vector ν forms the sequence of every other Jacobsthal number. That is,

$$\nu = \frac{1}{\sum_{k=1}^{n} J_{2k}}(J_{2n}, \ldots. J_6, J_4, J_2).$$

PROOF.

For the above transition matrix, let the limiting probability vector be given as

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

Solving for ν, we have the balance equation

$$\nu_k = b\nu_{k-1} + (1-5b)\nu_k + 4b\nu_{k+1},$$

$1, 2, \ldots, n-2$. That is,

$$5\nu_k = \nu_{k-1} + 4\nu_{k+1}. \tag{27}$$

Reversal together with increasing the spread in the index mean that we must show

$$5J_k = J_{k+2} + 4J_{k-2}. \tag{28}$$

From the definition of Jacobsthal sequence,

$$J_k = J_{k-1} + 2J_{k-2}.$$

$$J_{k-1} = J_{k-2} + 2J_{k-3}.$$

Adding gives

$$J_k = 3J_{k-2} + 2J_{k-3}.$$

But

$$J_{k-3} = J_{k-2} - 2J_{k-4}.$$

This implies

$$J_k = 5J_{k-2} - 4J_{k-4}.$$

Equation (27) corresponds to (28) , and the initial condition is given by the last column. Hence the limiting probability vector forms the normalized sequence of every other Jacobsthal sequence number. □

DEFINITION 6.1.2. *Using Binet's closed form approach, Jacobsthal sequence numbers can be expressed as* [**16**]

$$J_n = \frac{1}{3}(2^n - (-1)^n). \quad n = 0, 1, 2, \dots .$$

THEOREM 6.1.3.

For the class of infinite state space transition matrices whose limiting probability vector forms the sequence of Jacobsthal numbers,

$$\nu = (\tfrac{3}{4}, \tfrac{3}{16}, \tfrac{3}{64}, \dots)$$

PROOF.

Using the Binet's closed form for Jacobsthal numbers,

$$J_n = \tfrac{1}{3}(2^n - (-1)^n).$$

Then,

$$\frac{J_{n-1}}{J_n} = lim_{n\to\infty}\frac{2^{n-1}-(-1)^{n-1}}{2^n-(-1)^n}.$$

Therefore,

$$lim_{n\to\infty}\frac{J_{n-1}}{J_n} = \frac{1-(\frac{-1}{2})^{n-1}}{2+(\frac{-1}{2})^{n-1}} = \frac{1}{2}.$$

Also

$$\frac{J_{2n}}{\sum_{k=1}^{n} J_{2k}} = \frac{2^{2n}-(-1)^{2n}}{\sum_{k=1}^{n}(2^{2k}-(-1)^{2k})}.$$

This can be simplified to

$$\frac{J_{2n}}{\sum_{k=1}^{n} J_{2k}} = \frac{2^{2n}-(-1)^{2n}}{\frac{2^{2n+2}}{3}+(n-\frac{4}{3})}.$$

Taking limits,

$$lim_{n\to\infty}\frac{J_{2n}}{\sum_{k=1}^{n} J_{2k}} = \frac{3}{4}.$$

Hence, the limit exists and it is less than 1.

For the infinite state space transition matrices,

$$\nu = (lim_{n\to\infty}\frac{J_{2n}}{\sum_{k=1}^{n} J_{2k}}, lim_{n\to\infty}\frac{J_{2n-2}}{\sum_{k=1}^{n} J_{2k}}, lim_{n\to\infty}\frac{J_{2n-4}}{\sum_{k=1}^{n} J_{2k}}, \dots).$$

Let $\sum_{k=1}^{n} J_{2k} = \Sigma$ and denote $\theta_1 = lim_{n\to\infty}\frac{J_{2n}}{\Sigma}$. Then,

$$\nu = (\theta_1, \theta_1\alpha^2, \theta_1\alpha^4, \dots),$$

But

$$\theta_1 + \theta_1\alpha^2 + \theta_1\alpha^4 + \dots = 1.$$
$$\theta_1 = 1-\alpha^2 = \tfrac{3}{4},$$

and the result is proved. □

It will be noticed that for the infinite state space, the process represents a Birth-Death process whose rate matrix is

$$\begin{pmatrix} -b & b & . & . & . & . & . \\ 4b & -5b & b & . & . & . & . \\ 0 & 4b & -5b & b & . & . & . \\ 0 & 0 & 4b & -5b & b & . & . \\ . & . & . & . & . & .. & . \end{pmatrix}.$$

where $b > 0$. An example of this type of a system is a queueing system whose service rate is four times the arrival rate, i.e. M/M/1 where $\mu = 4\lambda$.

6.2. Pell Numbers

The Pell sequence [**16**] is a sequence of integers whose initial values are $p_0 = 0$, $p_1 = 1$. The sequence is governed by the recursive relation

$$p_n = 2p_{n-1} + p_{n-2}. \quad n = 2, 3, \ldots .$$

The first few terms of the sequence are:

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots .$$

We consider the general form of the class of transition matrices whose limiting probability vector forms normalized sequences of Pell numbers.

THEOREM 6.2.1.

For $0 < b \leq \frac{1}{6}$, *we consider the class of* $n \times n$ *transition matrices of the form*

$$\begin{pmatrix} 1-b & b & 0 & . & . & . & 0 & 0 & 0 \\ 5b & 1-6b & b & 0 & . & . & 0 & 0 & 0 \\ 4b & b & 1-6b & b & 0 & . & 0 & 0 & 0 \\ 4b & 0 & b & 1-6b & b & . & 0 & 0 & 0 \\ 4b & 0 & 0 & b & 1-6b & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 4b & 0 & . & . & . & . & 1-6b & b & 0 \\ 4b & 0 & . & . & . & . & b & 1-6b & b \\ 4b & 0 & . & . & . & . & 0 & b & 1-5b \end{pmatrix}.$$

The corresponding limiting probability vector ν forms the normalized sequence of "every other" Pell number. That is,

$$\nu = \frac{1}{\sum_{k=1}^{n} p_{2k-1}}(p_{2n-1}, p_{2n-3}, \ldots p_5; p_3, p_1).$$

PROOF.

For the above transition matrix, let the limiting probability vector be

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

Solving for ν, we have the balance equation

$$\nu_k = b\nu_{k-1} + (1-6b)\nu_k + b\nu_{k+1}, \quad k = 1, 2, \ldots, n-2.$$

That is,

$$6\nu_k = \nu_{k-1} + 4\nu_{k+1}. \tag{29}$$

Using reversal and increasing the index spread by a factor of 2 mean, we must show that

$$6p_k = p_{k+2} - 4p_{k-2}, \quad k = 2, 3, \ldots. \tag{30}$$

From the definition of the Pell sequence,

$$p_k = 2p_{k-1} + p_{k-2},$$

$$p_{k-1} = 2p_{k-2} + p_{k-3},$$

Hence,

$$p_k = 5p_{k-2} + 2p_{k-3}.$$

But

$$2p_{k-3} = p_{k-2} - p_{k-4}.$$

This implies

$$p_k = 6p_{k-2} - p_{k-4}.$$

This corresponds to (30), and the initial condition is given by the last column. Hence the limiting probability vector forms the normalized sequence of every other Pell sequence number. □

6.3. Every other Fibonacci Number

We consider the sequence of integers governed by the recursive relation

$$S_n = 3S_{n-1} - S_{n-2}; \quad n = 2, 3, \dots . \tag{31}$$

and the initial values $S_0 = 0$, and $S_1 = 1$. The first few terms of the sequence are:

$$0, 1, 3, 8, 21, 55, 144, 377, \dots .$$

We recognize these as the sequence of even indexed Fibonacci numbers. We present the following result concerning the class of matrices whose limiting probability vector forms the sequence of even indexed Fibonacci numbers.

THEOREM 6.3.1.

For $0 < b \leq \frac{1}{8}$, consider the class of transition matrices of the form

$$\begin{pmatrix} 1-3b & 3b & 0 & 0 & . & . & 0 & 0 \\ 5b & 1-8b & 3b & 0 & . & . & 0 & 0 \\ 5b & 0 & 1-8b & 3b & . & . & 0 & 0 \\ 4b & b & 0 & 1-8b & . & . & 0 & 0 \\ . & . & . & .. & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 4b & 0 & 0 & . & . & 0 & 1-8b & 3b \\ 2b & 0 & 0 & . & . & b & 0 & 1-3b \end{pmatrix}.$$

The corresponding limiting probability vector ν forms the normalized sequence of even indexed Fibonacci numbers. That is,

$$\nu = \frac{1}{\sum_{k=1}^{n} S_{2k}}(S_{2n}, S_{2n-2}, ..., S_6, S_4, S_2).$$

PROOF.

For the above transition matrix, let the limiting probability vector be given as

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

Solving for ν, we have the balance equation

$$\nu_k = 3b\nu_{k-1} + (1-8b)\nu_k + b\nu_{k+2},$$

$k = 1, 2, \ldots, n-3$. That is,

$$8\nu_k = 3\nu_{k-1} + \nu_{k+2}. \tag{32}$$

Using reversal means that we must show that

$$8S_k = 3S_{k+1} + S_{k-2}. \tag{33}$$

From the definition of the above sequence,

$$S_k = 3S_{k-1} - S_{k-2},$$

$$3S_{k-2} = S_{k-1} + S_{k-3}.$$

Hence,

$$S_k = 3S_{k-1} - \tfrac{1}{3}(S_{k-1} + S_{k-3}).$$

Therefore,

$$S_k = \tfrac{8}{3}S_{k-1} - \tfrac{1}{3}S_{k-3}.$$

This implies

$$3S_k = 8S_{k-1} - S_{k-3}.$$

This matches equation (33). The initial condition is given by the last column. Hence the limiting probability vector forms the normalized sequence of even indexed Fibonacci numbers. □

From Theorem 6.3.1, the matrix in Theorem 3.02 can be modified by changing (2b) in the last row to 3b. This results in a transition matrix whose limiting probability vector is the normalized sequence of even-indexed Fibonacci numbers. Hence, we have another class of transition matrices whose limiting probability vector forms the normalized sequence of even indexed Fibonacci numbers. This confirms the proposition that the class of transition matrices whose limiting probability vector forms Fibonacci numbers is not unique. We therefore examine other properties of our new class of transition matrices.

Since even indexed Fibonacci numbers [**9**],

$$F_2 + F_4 + \dots + F_{2n-2} + F_{2n} = F_{2n+1} - 1$$

we have the following result.

COROLLARY 6.3.2.

The limiting probability vector of the class of transition matrices from Theorem (6.3.1) is given as

$$\nu = \tfrac{1}{F_{2n+1}-1}(F_{2n}, F_{2n-2}, \dots, F_4, F_2).$$

Hence, for the infinite state space, the limiting probability vector for this class of transition matrices is the same as the one in Theorem 4.3. Extending this approach,we have the following result about every fourth Fibonacci numbers.

THEOREM 6.3.3.

For $0 < b \leq \frac{1}{8}$, *the class of transition matrices of the form*

$$\begin{pmatrix} 1-b & b & 0 & . & . & . & 0 & 0 & 0 \\ 6b & 1-7b & b & 0 & . & . & 0 & 0 & 0 \\ 5b & b & 1-7b & b & . & . & 0 & 0 & 0 \\ 5b & 0 & b & 1-7b & b & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 5b & 0 & . & . & . & . & b & 1-7b & b \\ 7b & 0 & . & . & . & . & . & b & 1-8b \end{pmatrix}$$

has the limiting probability vector ν *of the form*

$$\nu = \frac{1}{\sum_{k=1}^{n} S_{4k-2}}(S_{4n-2}, S_{4n-6}, ..., S_{10}, S_6, S_2).$$

PROOF.

For the above transition matrix, let the limiting probability vector be given as

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

Solving for ν, we have the balance equation

$$\nu_k = b\nu_{k-1} + (1-7b)\nu_k + b\nu_{k+1}.$$

$k = 1, 2, \ldots n-2$. That is,

$$7\nu_k = \nu_{k-1} + \nu_{k+1}. \tag{34}$$

Reversing the order and increasing the spread by a factor of 2 mean that we must

$$7S_k = S_{k+2} + S_{k-2}. \quad k = 2, 3, .. \tag{35}$$

From the definition of the above sequence,

$$S_k = 3S_{k-1} - S_{k-2},$$
$$S_{k-1} = 3S_{k-2} - S_{k-3}.$$

Thus,

$$S_k = 8S_{k-2} - 3S_{k-3}.$$

But

$$3S_{k-3} = S_{k-2} + S_{k-4}.$$

Therefore,

$$S_k = 7S_{k-2} - S_{k-4}.$$

This corresponds to (35). The initial condition is given by the last column. Hence the limiting probability vector forms the normalized sequence of every fourth Fibonacci number. □

COROLLARY 6.3.4.

Since for every fourth Fibonacci number [9]

$$F_2 + F_6 + F_{10} + ... + F_{4n-6} + F_{4n-2} = F_{2n}^2,$$

then ν *can be simplified as*

$$\nu = \frac{1}{F_{2n}^2}(F_{4n-2}, F_{4n-6}, ..., F_6, F_2).$$

THEOREM 6.3.5.

For the infinite state version of the class of finite transition matrices whose limiting probability vector forms the normalized sequence of every fourth Fibonacci number, ν *becomes*

$$\nu = (1 - \ell^4, \ell^4 - \ell^8, \ell^8 - \ell^{12}, ...),$$

where ℓ *the inverse of the golden ratio.*

PROOF.

For the finite $n \times n$ transition matrix given in Theorem 6.3.3,

$$\nu = \frac{1}{F_{2n}^2}(F_{4n-2}, F_{4n-6}, ..., F_6, F_2).$$

For the infinite state space case,

$$\nu = (lim_{n \to n\infty} \frac{F_{4n-2}}{\sum_{i=1}^n F_{4k-2}}, lim_{n \to n\infty} \frac{F_{4n-6}}{\sum_{i=1}^n F_{4k-2}}, \ldots).$$

Let $\Sigma = \sum_{i=1}^n F_{4k-2}$, and $\varepsilon = lim_{n \to n\infty} \frac{F_{4n-2}}{\Sigma}$. Then,

$$\nu = (\varepsilon, \varepsilon\ell^4, \varepsilon\ell^8, \ldots).$$

But

$$\varepsilon(1 + \ell^4 + \ell^8 +, ...) = 1,$$
$$\varepsilon = 1 - \ell^4,$$

and the result is proved. □

Still of interest to us is the method of finding the transition matrices yielding every nth number of the Fibonacci sequence, where $n \geq 3$ in the limiting vector.

6.4. Geometric Sequences

In this section and later sections, we still use the notation F_n but it may take meanings other than the Fibonacci numbers. We consider another class of sequences whose initial value is $F_1 = 1$. This class of sequences is governed by the recursive relation:

$$F_n = kF_{n-1}, \qquad n = 2, 3, \ldots$$

where $k > 1$ is fixed.

We present a general form for the resulting classes of transition matrices whose limiting vector forms normalized geometric sequences.

THEOREM 6.4.1.

For $0 < b \leq \frac{1}{k}$, where $k > 1$,the class of transition matrices for the form

$$\begin{pmatrix} 1-b & b & 0 & 0 & .. & 0 & 0 \\ (k-1)b & 1-kb & b & 0 & .. & 0 & 0 \\ (k-1)b & 0 & 1-kb & b & .. & 0 & 0 \\ (k-1)b & 0 & 0 & 1-kb & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & . \\ (k-1)b & 0 & .. & .. & .. & 1-kb & b \\ kb & 0 & .. & .. & .. & 0 & 1-kb \end{pmatrix}$$

has the limiting probability vector

$$\nu = \frac{1}{\sum_{i=1}^{i=n} k^{i-1}}(k^{n-1}, k^{n-2}, ..., k^2, k, 1).$$

The proof follows from the usual pattern. Since $\sum_{i=1}^{n} k^{i-1} = \frac{1-k^n}{1-k}$, we have the following corollary.

COROLLARY 6.4.2.

The limiting probability vector of the above class of matrices can be written as

$$\nu = \frac{k-1}{k^n-1}(k^{n-1}, k^{n-2}, ...k^2, k, 1).$$

THEOREM 6.4.3.

For the infinite state version of the transition matrix in theorem (6.4.1), the limiting probability vector becomes

$$\nu = (\tfrac{k-1}{k}, \tfrac{k-1}{k^2}, \tfrac{k-1}{k^3}, \ldots).$$

PROOF.

For the class of $n \times n$ transition matrices given in 6.4.1,

$$\nu = \frac{1}{\sum_{i=0}^{n} k^i}(k^{n-1}, k^{n-2}, ..., k^2, k, 1).$$

Define $\Sigma = \sum_{i=0}^{n-1} k^i$ and $\alpha = lim_{n\to\infty}\frac{k^{n-1}}{\sum_{i=0}^{n-1} k^i}$. Then the limiting probability vector becomes

$$\nu = (\alpha, \tfrac{\alpha}{k}, \tfrac{\alpha}{k^2}, \tfrac{\alpha}{k^3}, \dots).$$

But

$$\alpha + \tfrac{\alpha}{k} + \tfrac{\alpha}{k^2} + \tfrac{\alpha}{k^3} + \cdots = 1.$$

Therefore,

$$\alpha = \tfrac{k-1}{k},$$

and the result is proved. □

6.5. Sequence of All Integers

Consider the sequence of all positive integers whose initial values are $F_0 = 0$, $F_1 = 1$ and the sequence is governed by the relation

$$F_n = 2F_{n-1} - F_{n-2}. \quad n = 2, 3, \dots . \tag{36}$$

The resulting sequence is $\{F_i\}_{i=1}^{n} = \{1, 2, 3, \dots\}$. We present a class of transition matrices whose limiting probability vector is the normalized sequence of positive integers.

THEOREM 6.5.1.

(a) *For $0 < b \leq \frac{1}{4}$,the class of transition matrices of the form*

$$\begin{pmatrix} 1-2b & 2b & 0 & . & . & 0 & 0. & 0 \\ b & 1-3b & 2b & 0 & . . & 0 & 0 & 0 \\ b & 0 & 1-3b & 2b & . . & 0 & 0 & 0 \\ 0 & b & 0 & 1-3b & . . & 0 & 0 & 0 \\ . & . & . & . & . . & . & . & . \\ . & . & . & . & . . & . & . & . \\ . & . & . & . & . . & 1-3b & 2b & 0 \\ 0 & 0 & . & . & . . & 0 & 1-3b & 2b \\ 3b & 0 & . & . & . . & b & 0 & 1-4b \end{pmatrix}$$

has the limiting probability vector of the normalized sequence of positive integers. That is,

$$\nu = \frac{1}{\sum_{i=1}^{i=n} i}(n, n-1, n-2, \ldots, 1) = \frac{2}{n(n+1)}(n, n-1, n-2, \ldots, 1).$$

(b) *For the infinite state case, the limiting probability vector does not exist.*

PROOF.

(a) For the above transition matrix, let the limiting probability vector be given as

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

Solving for ν, we have the balance equation

$$\nu_k = 2b\nu_{k-1} + (1-3b)\nu_k + b\nu_{k+2}; \quad k = 1, 2, \ldots, n-3.$$

That is,

$$3\nu_k = 2\nu_{k-1} + \nu_{k+2}. \tag{37}$$

From the reversal method, we must prove that

$$3F_k = 2F_{k+1} + F_{k-1}. \tag{38}$$

But from the definition in (36),

$$F_k = 2F_{k-1} - F_{k-2}.$$
$$2F_{k-2} = F_{k-1} + F_{k-3}.$$

Thus,

$$F_k = 2F_{k-1} - \tfrac{1}{2}(F_{k-1} + F_{k-3}),$$
$$F_k = \tfrac{3}{2}F_{k-1} - \tfrac{1}{2}F_{k-3}.$$

Therefore,

$$2F_{k+1} = 3F_k - F_{k-2}.$$

This corresponds to (38), and the initial condition is given by the last column. Hence the limiting probability vector is the normalized sequence of all integers up to n.

For the sequence of integers, we know

$$1 + 2 + 3 + ... + n = \tfrac{n(n+1)}{2}.$$

Therefore, the limiting probability vector is

$$\nu = \tfrac{1}{\sum_{i=1}^{i=n} i}(n, n-1, n-2, \ldots, 1) = \tfrac{2}{n(n+1)}(n, n-1, \ldots, 2, 1).$$

(b) Since $lim_{n\to\infty} \frac{2n}{n(n+1)} = 0$, the limiting probability vector does not exist for the infinite state space in this case.

□

THEOREM 6.5.2.

For $0 < b \leq \frac{1}{3}$, the class of $n \times n$ transition matrices with the same pattern as the 6×6 transition matrix below

$$\begin{pmatrix} 1-b & b & 0 & 0 & .. & 0 & 0 \\ b & 1-2b & b & 0 & .. & 0 & 0 \\ 0 & b & 1-2b & b & .. & 0 & 0 \\ 0 & 0 & b & 1-2b & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & .. & .. & .. & 1-2b & b \\ 2b & 0 & .. & .. & .. & b & 1-3b \end{pmatrix}$$

has the limiting probability vector

$$\nu = \frac{1}{\sum_{i=1}^{i=n}(2i-1)}(2n-1, 2n-3, ..., 5, 3, 1).$$

That is, ν forms the normalized sequence of odd integers.

PROOF.

For the above transition matrix, let the limiting probability vector be given as

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

Solving for ν, we have the balance equation

$$\nu_k = b\nu_{k-1} + (1-2b)\nu_k + b\nu_{k+1}, \quad k = 1, 2, \ldots, n-2.$$

That is,

$$2\nu_k = \nu_{k-1} + \nu_{k+1}. \tag{39}$$

Using reversal and doubling the index gap means that we must prove

$$2F_k = F_{k+2} + F_{k-2}. \tag{40}$$

From the definition in (36),

$$F_k = 2F_{k-1} - F_{k-2},$$
$$F_{k-1} = 2F_{k-2} - F_{k-3},$$

so

$$F_k = 3F_{k-2} - 2F_{k-3}.$$

Therefore,

$$F_k = 2F_{k-2} - F_{k-4}.$$

This corresponds to (40). The initial condition is given by the last column. Hence, the limiting probability vector forms the normalized sequence of odd integers. □

In this case, it will also be noticed that in the infinite state space, the system forms a birth and death process with $\lambda = \mu$. The stability condition for birth and death system is not satisfied and therefore the limiting probability vector does not exist.

CHAPTER 7

General Results

In this section, we present some results regarding the general form for some classes of transition matrices whose limiting probability vectors form various classes of sequences. First, we present the general form of the class of transition matrices whose limiting probability is the normalized class of sequences whose initial values are 0 and 1 and the multipliers are k and $k-1$. We later extend this to the class of Horadam sequences, with general initial values and general multipliers. Again, F_i does not refer to Fibonacci numbers in this chapter.

THEOREM 7.0.3.

For fixed k from $\{2, 3, \dots\}$, consider the class of sequences governed by the relation

$$F_n = kF_{n-1} - (k-1)F_{n-2} \tag{41}$$

where $F_0 = 0$, $F_1 = 1$. Let $A = k^2 - k + 1$. Then, for $0 < b \leq min\{\frac{1}{A}, \frac{1}{k^2}\}$, the class of $n \times n$ transition matrices of the form

$$\begin{pmatrix} 1-kb & kb & 0 & .. & .. & .. & 0 & 0 \\ (k-1)^2b & 1-Ab & kb & 0 & .. & .. & 0 & 0 \\ (k-1)^2b & 0 & 1-Ab & kb & .. & .. & 0 & 0 \\ 0 & (k-1)^2 & 0 & 1-Ab & .. & .. & 0 & 0 \\ 0 & 0 & (k-1)^2b & .. & .. & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & . \\ 0 & 0 & .. & .. & .. & 0 & 1-Ab & kb \\ (2k-1)b & 0 & .. & .. & .. & (k-1)^2b & 0 & 1-k^2b \end{pmatrix}$$

has the limiting probability vector ν of the normalized sequence of every number of the sequence. That is,

$$\nu = \frac{1}{\sum_{i=1}^{n} F_i}(F_n, F_{n-1}, ...F_2, F_1).$$

PROOF.

For the sequence governed by the above recursive relation, the general form of the first few terms of the sequence is given as

$$0, 1, k, (k^2 - k + 1), [k(k^2 - k + 1) - k(k - 1)], \ldots .$$

Let the limiting probability vector ν be given as

$$\nu = (\nu_0, \nu_1, ..., \nu_{n-2}, \nu_{n-1}).$$

Then, for $i = 1, \ldots, n-3$, the balance equation is given as

$$kb\nu_{i-1} + [1 - (k^2 - k + 1)b]\nu_i + (k-1)^2 b\nu_{i+2} = \nu_i.$$

That is,

$$(k^2 - k + 1)\nu_i = k\nu_{i-1} + (k-1)^2\nu_{i+2}. \tag{42}$$

By reversal, we must show that

$$(k^2 - k + 1)F_i = kF_{i+1} - (k-1)^2 F_{i-2}. \tag{43}$$

From the definition of the sequence in (41),

$$F_i = kF_{i-1} - (k-1)F_{i-2}.$$

$$kF_{i-2} = F_{i-1} + (k-1)F_{i-3}, \text{ so}$$

$$F_i = kF_{i-1} - \frac{(k-1)}{k}[F_{i-1} + (k-1)F_{i-3}].$$

Therefore,

$$kF_i = (k^2 - k + 1)F_{i-1} - (k-1)^2 F_{i-3}.$$

Note this equation corresponds to equation (43). The last column is different from the others. The adjustment in the last row is based on the probabilistic property of transition matrices that the rows sum to 1. The adjustments observed are for the

$(n-1, n-1)$, $(n-1, 0)$ entries of the transition matrix. Hence, the limiting probability vector forms the normalized sequence of the sequence described in (41). □

For this class of transition matrix and sequence, we investigate the behavior of the limiting vector for the infinite state space.

THEOREM 7.0.4.

For the class of inifinite state transition matrices extended from the finite state case of Theorem 7.0.3, the limiting probability vector for the infinite state space is given as

$$\nu = \left(\tfrac{k-2}{k-1}, \tfrac{k-2}{(k-1)^2}, \tfrac{k-2}{(k-1)^3}, \dots\right),$$

where $k > 2$ is fixed.

PROOF.

If $\rho = lim_{n\to\infty} \frac{F_{n-1}}{F_n}$ (see Appendix A), then from the recursive relation (41)

$$1 = k\tfrac{F_{n-1}}{F_n} - (k-1)\tfrac{F_{n-2}}{F_n}.$$

Taking limits, we have,

$$1 = k\rho - (k-1)\rho^2.$$

Therefore,

$$\rho = \frac{-k \pm \sqrt{k^2 - 4(k-1)}}{-2(k-1)}.$$

Simplifying gives $\rho = \frac{1}{k-1}$ as the only positive real root that is less than 1. We work with this positive value of ρ. Hence for the infinite state case, ν becomes

$$\nu = (lim_{n\to\infty} \tfrac{F_n}{\sum_{r=1}^{n} F_r}, lim_{n\to\infty} \tfrac{F_{n-1}}{\sum_{r=1}^{n} F_r}, \dots).$$

Define $\gamma = lim_{n\to\infty} \frac{F_n}{\sum_{r=1}^{n} F_r}$ (see Appendix A). Then

$$\nu = (\gamma, \gamma\rho, \gamma\rho^2, \dots).$$

But $\gamma + \gamma\rho + \gamma\rho^2 + ... = 1$, so

$$\gamma = 1 - \rho = 1 - \tfrac{1}{k-1} = \tfrac{k-2}{k-1},$$

and the result is proved. $\square$

THEOREM 7.0.5.

Let $A = k^2 - 2k + 2$, *then for* $0 < b \leq \frac{1}{k^2-k+1}$, *the class of* $n \times n$ *transition matrices of the form*

$$\begin{pmatrix} 1-b & b & 0 & 0 & .. & 0 & 0 \\ (k-1)^2 b & 1-Ab & b & 0 & .. & 0 & 0 \\ 0 & (k-1)^2 b & 1-Ab & b & .. & 0 & 0 \\ 0 & 0 & (k-1)^2 b & 1-Ab & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & 1-Ab & b \\ kb & 0 & 0 & 0 & .. & (k-1)^2 b & 1-(k^2-k+1)b \end{pmatrix}$$

has the limiting probability vector ν *of the normalized sequence of "every other" number of the Horadam sequence with the initial values 0, 1 and multipliers k, (k-1); namely*

$$\nu = \tfrac{1}{\sum_{r=1}^{r=n} F_{2r-1}}(F_{2n-1}, F_{2n-3}, ..., F_3, F_1).$$

PROOF.

Let the limiting probability vector ν be given as

$$\nu = (\nu_0, \nu_1, ..., \nu_{n-2}, \nu_{n-1}).$$

For $i = 1, \ldots, n-2$, the balance equation is given as

$$b\nu_{i-1} + [1 - (k^2 - 2k + 2)b]\nu_i + (k-1)^2 b\nu_{i+2} = \nu_i.$$

That is,

$$(k^2 - 2k + 2)\nu_i = \nu_{i-1} + (k-1)^2 \nu_{i+2}. \tag{44}$$

Using reversal, and increasing index spread by a factor of 2, we must show that

$$(k^2 - 2k + 2)F_i = F_{i+2} + (k-1)^2 F_{i-4}. \tag{45}$$

From the definition of the sequence in (41),

$$F_i = kF_{i-1} - (k-1)F_{i-2},$$
$$F_{i-1} = kF_{i-2} - (k-1)F_{i-3}, \text{ so}$$
$$F_i = k(kF_{i-2} - (k-1)F_{i-3}) - (k-1)F_{i-2}.$$

But $kF_{i-3} = F_{i-2} + (k-1)F_{i-4}$, therefore,

$$F_i = (k^2 - 2k + 2)F_{i-2} - (k-1)^2 F_{i-4}.$$

It will be observed that this equation corresponds to equation (45). The initial condition is accounted for in the last column. Hence, the limiting probability vector forms the normalized sequence of every other number of the sequence. □

THEOREM 7.0.6.

For the infinite state space case of the class of finite transition matrices presented in Theorem 7.0.5, the limiting probability vector is

$$\nu = [\tfrac{k^2-2k}{(k-1)^2}, \tfrac{k^2-2k}{(k-1)^4}, \tfrac{k^2-2k}{(k-1)^6}, \ldots].$$

PROOF.

For the finite version, the limiting probability vector is given as

$$\nu = \tfrac{1}{\sum_{r=1}^{r=n} F_{2r-1}}(F_{2n-1}, F_{2n-3}, \ldots, F_3, F_1).$$

Then, define $\omega = lim_{n\to\infty} \frac{F_{2n-1}}{\sum_{r=1}^{r=n} F_{2r-1}}$ (see appendix A for the existence of the limit). For the infinite state space transition matrix,

$$\nu = (\omega, \omega\rho^2, \omega\rho^4, \ldots).$$

But $\omega + \omega\rho^2 + \omega\rho^4 + \ldots = 1$. So,

$$\omega = 1 - \rho^2.$$

From the proof of theorem (7.0.4), we find $\rho = \frac{1}{k-1}$. Thus,

$$\omega = 1 - \tfrac{1}{(k-1)^2} = \tfrac{k^2-2k}{(k-1)^2},$$

and the result is proved. □

Next, we consider the sequence of integers governed by the relation

$$F_n = kF_{n-1} + (k-1)F_{n-2} \quad k = 2, 3, \ldots, \tag{46}$$

where $F_0 = 0$, $F_1 = 1$ and k is fixed. We present the class of transition matrices whose limiting probability vector forms the normalized sequence class of sequence shown in (7.0.5). The infinite state version is also considered.

THEOREM 7.0.7.

Let $A = k^2 + k - 1$, then for $0 < b \leq \frac{1}{A}$, the class of $n \times n$ transition matrices of the form

$$\begin{pmatrix} 1-kb & kb & 0 & .. & 0 & 0 & 0 & 0 \\ (A-k)b & 1-Ab & kb & .. & 0 & 0 & 0 & 0 \\ (A-k)b & 0 & 1-Ab & .. & 0 & 0 & 0 & 0 \\ (2k-2)b & (k-1)^2 & 0 & .. & 0 & 0 & 0 & 0 \\ (2k-2)b & 0 & (k-1)^2b & .. & .. & .. & . & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. \\ (2k-1)b & 0 & .. & .. & (k-1)^2b & (k-2)b & 1-Ab & b \\ (k-1)b & 0 & .. & .. & 0 & 0 & b & 1-kb \end{pmatrix}$$

has the limiting probability vector ν of the normalized sequence of every number of the Horadam sequence in (46). That is,

$$\nu = \tfrac{1}{\sum_{i=1}^{n} F_r}(F_n, F_{n-1}, \ldots F_2, F_1).$$

PROOF.

For the sequence governed by the above recursive relation, the first few terms of the sequence is given as

$$0, 1, k, (k^2 + k - 1), [k(k^2 + k - 1) + k(k - 1)], \ldots.$$

Let the limiting probability vector ν be given as

$$\nu = (\nu_0, \nu_1, ..., \nu_{n-2}, \nu_{n-1}).$$

For $i = 1, \ldots, n - 3$, the balance equation is given as

$$kb\nu_{i-1} + [1 - (k^2 + k - 1)b]\nu_i + (k - 1)^2 b\nu_{i+2} = \nu_i.$$

That is,

$$(k^2 + k - 1)\nu_i = k\nu_{i-1} + (k - 1)^2 \nu_{i+2}. \tag{47}$$

Using reversal, we must show that

$$(k^2 + k - 1)F_i = kF_{i+1} - (k - 1)^2 F_{i-2}. \tag{48}$$

From the definition of the sequence in (46),

$$F_i = kF_{i-1} + (k - 1)F_{i-2}.$$
$$kF_{i-2} = F_{i-1} - (k - 1)F_{i-3}, \text{ so}$$
$$F_i = kF_{i-1} + \tfrac{(k-1)}{k}[F_{i-1} - (k - 1)F_{i-3}],$$

Therefore,

$$kF_i = (k^2 + k - 1)F_{i-1} - (k - 1)^2 F_{i-3}.$$

Note that this equation corresponds to equation (48) and that the last two rows are different from the others. The adjustment in the last two rows is based on the initial values of the sequence and the probabilistic property of transition matrices that each row sum to 1. The adjustments observed are for the $(n - 1, n - 1)$, $(n - 2, 0)$, and $(n - 2, n - 3)$ entries of the transition matrices. Hence, the limiting probability vector is the normalized sequence of every number of the sequence in (46). □

THEOREM 7.0.8.

For the class of transition matrices presented in Theorem 7.0.7, the limiting probability vector for the infinite state space is given as

$$\nu = (1-\rho, (1-\rho)\rho, (1-\rho)\rho^2, \dots),$$

where ρ is the positive real root (less than 1) of the equation

$$(k-1)\rho^2 + k\rho - 1 = 0.$$

PROOF.

Let $\rho = lim_{n\to\infty}\frac{F_{n-1}}{F_n}$ (see Appendix A for proof). From (46),

$$1 = k\frac{F_{k-1}}{F_n} + (k-1)\frac{F_{n-2}}{F_n}.$$

Taking limits,

$$k\rho + (k-1)\rho^2 - 1 = 0.$$

Then,

$$\rho = \frac{-k\pm\sqrt{k^2+4k-4}}{k-1}.$$

We work with the positive value of ρ that is less than 1. For the finite state case,

$$\nu = \frac{1}{\sum_{r=1}^{n} F_r}(F_n, F_{n-1}, F_{n-2}, ..., F_1).$$

. Define $\gamma = lim_{n\to\infty}\frac{F_n}{\sum_{r=1}^{n} F_r}$. Then for the infinite state space,

$$\nu = (\gamma, \gamma\rho, \gamma\rho^2, ...).$$

But $\gamma + \gamma\rho + \gamma\rho^2 + ... = 1$. So,

$$\gamma = 1 - \rho,$$

and the result is proved. □

We now present results on the class of transition matrices whose limiting probability vector form the normalized sequence of every other number of the sequence in (46)

THEOREM 7.0.9.

For $0 < b \leq \frac{1}{A}$, where $A = k^2 + 2k - 2$, the class of $n \times n$ transition matrices of the form

$$\begin{pmatrix} 1-b & b & 0 & 0 & .. & 0 & 0 \\ (k-1)^2b & 1-Ab & b & 0 & .. & 0 & 0 \\ 4(k-1)b & (k-1)^2b & 1-Ab & b & .. & .. & 0 \\ 4(k-1)b & 0 & (k-1)^2b & 1-Ab & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 4(k-1)b & .. & .. & .. & .. & 1-Ab & b \\ kb & 0 & & .. & .. & (k-1)^2b & 1-(k^2-k+1)b \end{pmatrix}$$

has the limiting probability vector ν of the normalized sequence of "every other" number of the sequence (46),

$$\nu = \frac{1}{\sum_{r=1}^{r=n} F_{2r-1}}(F_{2n-1}, F_{2n-3}, ...F_3, F_1).$$

PROOF.

Let the limiting probability vector ν be given as

$$\nu = (\nu_0, \nu_1, ..., \nu_{n-2}, \nu_{n-1}).$$

For $i = 1, \ldots, n-2$, the balance equation is given as

$$b\nu_{i-1} + [1 - (k^2 + 2k - 2)b]\nu_i + (k-1)^2 b\nu_{i+1} = \nu_i.$$

That is,

$$(k^2 + 2k - 2)\nu_i = \nu_{i-1} + (k-1)^2\nu_{k+1}. \tag{49}$$

Using reversal and increasing the index gap by a factor of 2 mean we must prove that

$$(k^2 + 2k - 2)F_i = F_{i+2} - (k-1)^2 F_{i-2}. \tag{50}$$

From the definition of the sequence in (46),

$$F_i = kF_{i-1} + (k-1)F_{i-2},$$

$$F_{i-1} = kF_{i-2} + (k-1)F_{i-3}, \text{ so}$$
$$F_i = k(kF_{i-2} + (k-1)F_{i-3}) + (k-1)F_{i-2}.$$

But $kF_{i-3} = F_{i-2} - (k-1)F_{i-4}$. Therefore,

$$F_i = (k^2 + 2k - 2)F_{i-2} - (k-1)^2 F_{i-4}.$$

It will be noticed that equation (49) matches equation (50). The initial condition is accounted for in the last column. Hence, the limiting probability vector forms the normalized sequence of every other number of the sequence. □

THEOREM 7.0.10.

For the infinite state transition matrices corresponding to, the class of transition matrices whose limiting probability vector forms the sequence of every other number in the sequence, the limiting probability vector is

$$\nu = ((1-\rho^2, \rho^2(1-\rho^2), \rho^4(1-\rho^2), \ldots),$$

where $\rho = \frac{-k+\sqrt{k^2+4k-4}}{2(k-1)}$.

PROOF. For the finite version, the limiting probability vector is given as

$$\nu = \frac{1}{\sum_{r=1}^{r=n} F_{2r-1}}(F_{2n-1}, F_{2n-3}, \ldots, F_3, F_1).$$

Define $\omega = lim_{n\to\infty}\frac{F_{2n-1}}{\sum_{r=1}^{r=n} F_{2r-1}}$ (see Appendix A for the justification of the existence of this limit). For the infinite state case,

$$\nu = (\omega, \omega\rho^2, \omega\rho^4, \ldots).$$

But $\omega + \omega\rho^2 + \omega\rho^4 + \ldots = 1$. Therefore,

$$\omega = 1 - \rho^2,$$

and the result is proved. □

7.1. Horadam Sequences

A.F Horadam [**17**] described a class of sequences where the initial values are $H_0 = j$, and $H_1 = i$. The resulting class of sequences is governed by the recursive relation

$$H_n = xH_{n-1} \pm yH_{n-2}. \quad n = 2, 3, \ldots . \tag{51}$$

The class of sequences thus formed is known as Horadam sequence, also written as $H(j, i, x, y)$.

In this section we present general results concerning this class of sequences. First we investigate the class of transition matrices whose limiting probability vector form the sequence of Horadam numbers where the initial values are 0 and 1, namely $H(0, 1, x, y)$ where $x \geq y > 0$. The theorems presented in this section include 2 cases simultaneously, namely $H_n = xH_{n-1} + yH_{n-2}$, and $H_n = xH_{n-1} - yH_{n-2}$.

THEOREM 7.1.1.

For $0 < b \leq min\{x, \frac{1}{x^2 \pm y}, \frac{1}{x-y^2}\}$, $x \geq y^2$, *and* $x^2 \pm y - x$, *the class of* $n \times n$ *transition matrices with the same pattern as the* 6×6 *transition matrix given below*

$$\begin{pmatrix} 1-xb & xb & 0 & 0 & 0 & 0 \\ (x^2 \pm y - x)b & 1-(x^2 \pm y)b & xb & 0 & 0 & 0 \\ (x^2 \pm y - x)b & 0 & 1-(x^2 \pm y)b & xb & 0 & 0 \\ (x^2 \pm y - y^2 - x)b & y^2 b & 0 & 1-(x^2 \pm y)b & xb & 0 \\ (x^2 \pm y - y^2 - 1)b & 0 & y^2 b & 0 & 1-(x^2 \pm y)b & b \\ (x - y^2)b & 0 & 0 & y^2 b & 0 & 1-xb \end{pmatrix},$$

has the limiting probability vector form the normalized sequence of every number of the sequence $\{H_k\}$.

In the above, the condition that the entries be probabilities is violated if $x < y^2$. *Instead, for* $x < y^2$, *the class of transition matrix given below is appropriate.*

$$\begin{pmatrix} 1-xb & xb & 0 & 0 & 0 & 0 \\ (x^2\pm y-x)b & 1-(x^2\pm y)b & xb & 0 & 0 & 0 \\ (x^2\pm y-x)b & 0 & 1-(x^2\pm y)b & xb & 0 & 0 \\ (x^2\pm y-y^2-x)b & y^2b & 0 & 1-(x^2\pm y)b & xb & 0 \\ (x^2\pm y-y^2-\gamma-1)b & 0 & y^2b & \gamma b & 1-(x^2\pm y)b & b \\ (x-r)b & 0 & 0 & rb & 0 & 1-xb \end{pmatrix},$$

where $y^2 = \gamma x + r$.

PROOF.

For any choice of x and y, the first few terms of the resulting sequence of integers are:

$$0, 1, x, x^2\pm y, x(x^2\pm y)\pm xy, \dots ,$$

where we have either 2 plus signs or 2 minus signs for the two cases. Let the limiting probability vector ν be given as

$$\nu = (\nu_0, \nu_1, ..., \nu_{n-2}, \nu_{n-1}).$$

For $i = 1, \dots, n-3$, the balance equation is given as

$$xb\nu_{i-1} + [1-(x^2 \pm y)b]\nu_i + y^2 b\nu_{i+2} = \nu_i.$$

That is,

$$(x^2 \pm y)\nu_i = x\nu_{i+1} + y^2\nu_{i+2}. \tag{52}$$

From reversal, we must show that

$$(x^2\pm y)H_i = xH_{i+1} + y^2H_{i-2}. \tag{53}$$

From the definition of the sequence in (51),

$$H_i = xH_{i-1}\pm yH_{i-2},$$

$$xH_{i-2} = H_{i-1}\mp yH_{i-3}.$$

Therefore,

$$xH_i = (x^2 \pm y)H_{i-1} - y^2 H_{i-3}.$$

This equation corresponds to equation (51). The variations in the last three columns of the matrices are due to the initial conditions of the sequence. Hence, the limiting probability vector forms the normalized sequence of every number of the Horadam sequence H_k. $\square$

THEOREM 7.1.2.

For $0 < b \leq \frac{1}{x^2 \pm 2y}$ and $y > 0$, the class of $n \times n$ transition matrices with the same pattern as the 6×6 transition matrix

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 \\ (x^2 \pm 2y - 1)b & 1-(x^2 \pm 2y)b & b & 0 & 0 & 0 \\ (x^2 \pm 2y - y^2 - 1)b & y^2 b & 1-(x^2 \pm 2y)b & b & 0 & 0 \\ (x^2 \pm 2y - y^2 - 1)b & 0 & y^2 b & 1-(x^2 \pm 2y)b & b & 0 \\ (x^2 \pm 2y - y^2 - 1)b & 0 & 0 & y^2 b & 1-(x^2 \pm 2y)b & b \\ (x^2 \pm y - y^2)b & 0 & 0 & 0 & y^2 b & 1-(x^2 \pm y)b \end{pmatrix}$$

has the limiting probability vector formed by the normalized sequence of every other number of the sequence . That is,

$$\nu = \frac{1}{\sum_{k=1}^{k=n} H_{2k-1}}(H_{2n-1}, H_{2n-3}, H_{2n-5}, ..., H_3, H_1).$$

PROOF.

Let the limiting probability vector ν be

$$\nu = (\nu_0, \nu_1, ..., \nu_{n-2}, \nu_{n-1}).$$

For $i = 1, \ldots, n-2$, the balance equations are given as

$$b\nu_{i-1} + [1 - (x^2 \pm 2y)b]\nu_i + y^2 b\nu_{i+1} = \nu_i.$$

That is,

$$(x^2 \pm 2y)\nu_i = \nu_{i-1} + y^2\nu_{i+1}. \tag{54}$$

Using reversal and increasing the index gap by a factor of 2 means we must show that

$$(x^2 \pm 2y)H_i = H_{i+2} - y^2 H_{i-2}. \tag{55}$$

From the definition of the sequence in (51),

$$H_i = xH_{i-1} \pm yH_{i-2},$$
$$H_{i-1} = xH_{i-2} \pm yH_{i-3}.$$

Therefore,

$$H_i = (x^2 \pm y)H_{i-2} \pm xyH_{i-3}.$$

But $xH_{i-3} = H_{i-2} \mp yH_{i-4}$, hence,

$$H_i = (x^2 \pm 2y)H_{i-2} - y^2 H_{i-4}.$$

Note that this corresponds to equation (55). The adjustment in the last column can be constructed by using the ratio of the fourth term to the second term of the sequence. □

7.2. General Results 2

In the previous section, we assumed initial values of 0 and 1. In this section, we present a more general result regarding the class of transition matrices whose limiting probability vector is formed by sequence(s) belonging to the Horadam class of sequences. We investigate the general class $H(j, i, x, y)$, namely, the class of sequence with the initial values $H_0 = j$ and $H_1 = i$, and the governing recursive relation

$$H_n = xH_{n-1} \pm yH_{n-2}, \quad n = 2, 3, \ldots . \tag{56}$$

where $x \geq y > 0$. We investigate the form of the class of transition matrices whose limiting probability vector is the normalized sequence of every(every other) number of the Horadam sequence for any choice of j, i, x, y for $x \geq y > 0$. The behavior of the limiting probability vector for the infinite state space is also investigated.

THEOREM 7.2.1.

Let $M = x^2 \pm 2y$, *then, for* $0 < b \leq min\{\frac{1}{M}, \frac{1}{y^2}, \frac{j}{xi+yj}\}$, $j > 0$, $M \geq y^2 + 1$, *and* $\frac{xi+yj}{j} \geq y^2$, *the class of transition probability matrices whose limiting probability vectors is the normalized sequence of "every other" number of the sequence in (56) is given by*

$$\begin{pmatrix} 1-b & b & 0 & \dots & & 0 & 0 & 0 \\ (M-1)b & 1-Mb & b & . & . & 0 & 0 & 0 \\ (M-y^2-1)b & y^2b & 1-Mb & b & . & 0 & 0 & 0 \\ (M-y^2-1)b & 0 & y^2b & 1-Mb & b \; .. & 0 & 0 & 0 \\ . & . & . & . & . \; . & . & . & . \\ . & . & . & . & . \; . & . & . & . \\ (M-y^2-1)b & 0 & . & . & . \; . & 1-Mb & b & 0 \\ (M-y^2-1)b & 0 & . & . & . \; . & y^2b & 1-Mb & b \\ (\frac{xi\pm yj}{j}-y^2)b & 0 & . & . & . \; . & 0 & y^2b & 1-\frac{(xi\pm yj)}{j}b \end{pmatrix}.$$

That is,

$$\nu = \frac{1}{\sum_{k=1}^{k=n} H_{2k}}(H_{2n}, H_{2n-2}, H_{2n-4}, ...H_2, H_0).$$

(b) For$0 < b \leq min\{\frac{1}{M}, \frac{1}{y^2}, \frac{i}{x(xi+yj)\pm yi}\}$, $M - y^2 - 1 \geq 0$ *and* $i > 0$, *the class of transition matrices given as*

$$\begin{pmatrix} 1-b & b & .. & .. & .. \; .. & .. & 0 \\ (M-1)b & 1-Mb & b & .. & .. \; .. & .. & 0 \\ (M-y^2-1)b & y^2b & 1-Mb & b & .. \; .. & .. & 0 \\ (M-y^2-1)b & 0 & y^2b & 1-Mb & b \; .. & 0 & 0 \\ .. & .. & .. & .. & .. \; .. & .. & . \\ .. & .. & .. & .. & .. \; .. & .. & . \\ (M-y^2-1)b & 0 & .. & .. & .. \; .. & 1-Mb & b \\ (\frac{x(xi\pm yj)\pm yi}{i}-y^2)b & 0 & .. & .. & .. \; .. & y^2b & 1-\frac{x(xi\pm yj)\pm yi}{i}b \end{pmatrix}$$

has the limiting probability vector of the normalized sequence of "every other" number of Horadam sequence gven in (56). That is,

$$\nu = \frac{1}{\sum_{k=1}^{k=n} H_{2k-1}}(H_{2n-1}, H_{2n-3}, H_{2n-5}, ...H_3, H_1).$$

PROOF.

Given the recursive relation in (56), the general form of the first few terms of the class sequences is given as:

$$j, i, xi{\pm}yj, x(xi{\pm}yj){\pm}yi, \dots . \tag{57}$$

Let the limiting probability vector

$$\nu = (\nu_0, \nu_1, ..., \nu_{n-2}, \nu_{n-1}).$$

For $i = 1, 2, \dots, n-2$, the balance equation is given as

$$b\nu_{i-1} + [1 - (x^2 \pm 2y)b]\nu_i + y^2 b\nu_{i+1} = \nu_i.$$

That is,

$$(x^2 \pm 2y)\nu_i = \nu_{i-1} + y^2\nu_{i+1}. \tag{58}$$

Using reversal and increasing the spread by a factor of 2 means that we must show that

$$(x^2{\pm}2y)H_i = H_{i+2} - y^2 H_{i-2}. \tag{59}$$

Iterating the terms of the sequence using the general recursive formula, we have

$$H_i = xH_{i-1}{\pm}yH_{i-2},$$
$$H_{i-1} = xH_{i-2}{\pm}yH_{i-3}.$$

Therefore,

$$H_i = (x^2 \pm y)H_{i-2}{\pm}xyH_{i-3}.$$

But $xH_{i-3} = H_{i-2}{\mp}yH_{i-4}$, so

$$H_i = (x^2{\pm}2y)H_{i-2} - y^2 H_{i-4}.$$

Equation (58) corresponds to equation (60). The last columns of the two transition matrices are formed by inserting the ratio of the fourth(third) term to the second(first) term of the sequence at the $(n-1, n-1)$ entry of the transition matrix. The $(n-1, 0)$ entry is found by adjusting the row to sum to 1. □

Hence, the above results give us the general form of the class of transition matrices whose limiting probability vector is formed by every other Horadam numbers for any choice of initial values. We now investigate the infinite state space behavior of the limiting probability vector.

THEOREM 7.2.2.

For the infinite state case, the class of transition matrices whose limiting probability vector is the normalized sequence of every other number of Horadam sequence has the limiting probability vector

$$\nu = (1 - L^2, (1 - L^2)L^2, (1 - L^2)L^4, ...)$$

where $0 < L \leq 1$ *is defined in the proof below.*

PROOF.

For infinite state space, let $L = lim_{n\to\infty} \frac{H_{n-1}}{H_n}$ (see Appendix A). Then taking the limit of both sides of the equation (56), we have

$$1 = lim_{n\to\infty}\{x\frac{H_{n-1}}{H_n} \pm y\frac{H_{n-2}}{H_n}\}.$$

That is,

$$1 = xL \pm yL^2$$

There are two cases:

(1)

$$1 = xL + yL^2$$

That is,

$$yL^2 + xL - 1 = 0.$$

$$L = \frac{-x \pm \sqrt{x^2 + 4y}}{2y}.$$

(2)

$$1 = xL - yL^2,$$

That is,

$$yL^2 - xL + 1 = 0.$$

$$L = \frac{x \pm \sqrt{x^2 - 4y}}{2y}$$

In both cases, we work with the appropriate value of L in (0,1). Therefore, for the class of transition matrices with the limiting probability vector

$$\nu = \frac{1}{\sum_{k=1}^{k=n} H_k}(H_{2n-1}, H_{2n-3}, H_{2n-5}, \ldots, H_3, H_1),$$

the limiting vector for the infinite state case is given as

$$\nu = (lim_{n\to\infty}\frac{H_{2n-1}}{\sum_{k=1}^{k=n} H_{2k-1}}, lim_{n\to\infty}\frac{H_{2n-3}}{\sum_{i=1}^{i=n} H_{2i-1}}, \ldots).$$

Let $\Sigma = \sum_{i=1}^{n} H_{2i-1}$ and let $\alpha = lim_{n\to\infty}\frac{H_{2n-1}}{\Sigma}$ (see Appendix A). Then

$$\nu = (\alpha, \alpha L^2, \alpha L^4, \alpha L^6, \ldots).$$

But

$$\alpha + \alpha L^2 + \alpha L^4 + \alpha L^6 + ... = 1.$$

Therefore, $\alpha = 1 - L^2$, and the result is proved. $\square$

THEOREM 7.2.3.

Let $N = x^2 \pm y$, then for $0 < b \leq min\{\frac{1}{N}, \frac{1}{x(xi \pm yj) \pm yi}, \frac{1}{xi \pm yj}\}$, $(xi \pm yj) \geq y^2$, and $(x^2 \pm y) \geq xi \pm yj + y^2$, the class of transition matrices of the form

$$\begin{pmatrix} 1-xb & xb & 0 & .. & .. & .. & .. & 0 \\ (N-x)b & 1-Nb & xb & 0 & .. & .. & .. & 0 \\ (N-x)b & 0 & 1-Nb & xb & .. & .. & .. & 0 \\ (N-y^2-x)b & y^2b & 0 & 1-Nb & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. & . \\ (N-y^2-(xi \pm yj))b & 0 & .. & .. & .. & 1-Nb & (xi \pm yj)b & 0 \\ \{x(xi \pm yj) \pm yi - y^2 - i\}b & 0 & .. & .. & .. & 0 & 1-(x(xi \pm yj) \pm yi)b & ib \\ (xi \pm yj - y^2)b & 0 & .. & .. & .. & y^2b & 0 & 1-(xi \pm yj)b \end{pmatrix}$$

has the limiting probability vector which is the normalized sequence of every Horadam sequence number. That is,

$$\nu = \frac{1}{\sum_{k=1}^{n} H_k}(H_n, H_{n-1}, H_{n-2}, \ldots, H_2, H_1). \tag{60}$$

PROOF.

Let the limiting probability vector be denoted by

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-2}, \nu_{n-1}).$$

Then, solving $\nu = \nu P$ gives balance equation

$$\nu_k = xb\nu_{k-1} + (1 - (x^2 \pm 2y))b\nu_k + y^2 b\nu_{k+2},$$

for $k = 2, 3, \ldots, n-3$. That is,

$$(x^2 \pm y)\nu_k = x\nu_{k-1} + y^2\nu_{k+2}. \tag{61}$$

Using reversal, we must show that

$$(x^2 \pm y)H_k = xH_{k-1} + y^2 H_{k-2}. \tag{62}$$

From the definition of Horadam sequences,

$$H_k = xH_{k-1} \pm yH_{k-2},$$
$$xH_{k-2} = H_{k-1} \mp yH_{k-3}.$$

Therefore,

$$xH_k = (x^2 \pm y)H_{k-1} - y^2 H_{k-3}.$$

It will be observed this corresponds to equation (62). In addition, given the general recursive relation given in (56), and the first few terms given in (57), the last two columns account for the first few terms of the sequence. Therefore, the limiting probability vector is formed by the normalized sequence of Horadam sequence numbers. $\square$

The above general result implies that the transition matrix can be formed as long as conditions are met. Hence with these general results, we now have classes of transition matrices whose limiting probability vectors is the normalized Horadam sequences (for $x \geq y > 0$). We illustrate these results with examples.

EXAMPLE 7.2.1.

For $x = 3, y = 2, i = 3$, *and* $j = 1$, *with the recursive formula (56), the first few terms of the resulting sequence are:*

$$1, 3, 11, 39, 139, 495, 1763, 6279, 22363, \ldots$$

For $0 < b \leq \frac{1}{13}$, *from Theorem (7.1.2), the class of* 5×5 *transition matrices*

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 \\ 12b & 1-13b & b & 0 & 0 \\ 8b & 4b & 1-13b & b & 0 \\ 8b & 0 & 4b & 1-7b & 3b \\ 7b & 0 & 0 & 4b & 1-11b \end{pmatrix}$$

has the limiting probability vector

$$\nu = \tfrac{1}{24277}(22363, 1763, 139, 11, 1)$$

which is the normalized sequence of "every other" number of the above sequence.

EXAMPLE 7.2.2.

Consider the sequence $H(0,1,3,-2)$*, with the first few terms :*

$$0, 1, 3, 7, 15, 31, 63, 127, 255, 511, \ldots .$$

To get the class of transition matrices whose limiting probability vector is the normalized sequence of the above sequence, we apply the result in Theorem (7.0.3).

For $0 < b \leq \frac{1}{7}$*, the class of* 7×7 *transition matrices*

$$\begin{pmatrix} 1-3b & 3b & 0 & 0 & 0 & 0 & 0 \\ 4b & 1-7b & 3b & 0 & 0 & 0 & 0 \\ 4b & 0 & 1-7b & 3b & 0 & 0 & 0 \\ 0 & 4b & 0 & 1-7b & 3b & 0 & 0 \\ 0 & 0 & 4b & 0 & 1-7b & 3b & 0 \\ b & 0 & 0 & 4b & b & 1-7b & b \\ 2b & 0 & 0 & 0 & b & 0 & 1-3b \end{pmatrix}$$

has the limiting probability vector

$$\nu = \tfrac{1}{247}(127, 63, 31, 15, 7, 3, 1),$$

which is the normalized sequence of every number of the above sequence.

The results presented in this chapter are summarized by a pictorial representation (See Appendix B).

CHAPTER 8

Horadam Sequences and Other Classes of Transition Matrices

In this section, other classes of transition matrices are presented whose limiting probability vector is the normalized sequence of Horadam sequences. Later in the section, some common properties of the classes of transition matrices are examined. The purpose is to give the general results regarding the classes of transition matrices examined in this work.

8.1. A New Class of Transition Matrices

As an example, we consider the transition matrix P for a Markov Chain with 5 states $\{0, 1, 2, 3, 4\}$ given as

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Then

$$\nu = \tfrac{1}{49}(29, 12, 5, 2, 1).$$

The limiting probability vector is the normalized sequence of the first 5 Pell Numbers (see section 6.2).

By uniformization, a corresponding rate matrix for this Markov chain is given as

$$\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

An example of a process that can be modelled by the above rate matrix is a queueing system where the customers arrive singly but are served either singly or in pairs. Consider an hospital where doctors attend to patients. The patients enter the system singly but are served either singly or in pairs (e.g. whenever a couple joins for treatment).

Extending this to transition matrices of higher finite dimensions gives more Pell numbers. The Pell numbers are a special case of Horadam numbers with $H(0, 1, 2, 1)$. A general form of the class of transition matrices with similar patterns have been discovered and the general result is provided below.

THEOREM 8.1.1.

For $0 < b \leq min\{\frac{1}{x+1}, \frac{i}{xi+yj}\}$, *and* $i > 0$, *the class of transition matrices of the form*

$$M = \begin{pmatrix} 1-b & b & 0 & 0 & . & 0 & 0 & 0 & 0 \\ xb & 1-(x+1)b & b & 0 & . & 0 & 0 & 0 & 0 \\ yb & (x-y)b & 1-(x+1)b & b & . & 0 & 0 & 0 & 0 \\ 0 & yb & (x-y)b & 1-(x+1)b & . & 0 & 0 & 0 & 0 \\ 0 & 0 & yb & (x-y)b & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & yb & (x-y)b & 1-(x+1)b & b \\ 0 & 0 & . & . & . & . & yb & [\frac{(x-y)i+yj}{i}]b & 1-[\frac{xi+yj}{i}]b \end{pmatrix}$$

has the limiting probability vector of the normalized sequence of every Horadam sequence number $H(j, i, x, y)$.

PROOF.

Let the limiting probability vector be

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

For $k = 1, 2, \ldots, n-3$, the balance equation for the above transition matrix is given by

$$\nu_k = b\nu_{k-1} + [1 - (x+1)b]\nu_k + (x-y)b\nu_{k+1} + yb\nu_{k+2},$$

that is,

$$(x+1)\nu_k = \nu_{k-1} + (x-y)\nu_{k+1} + y\nu_{k+2}. \tag{63}$$

Using reversal, we must show that

$$(x+1)H_k = H_{k+1} + (x-y)H_{k-1} + yH_{k-2}. \tag{64}$$

From the definition of the Horadam sequence numbers,

$$H_k = xH_{k-1} + yH_{k-2}.$$

Adding and subtracting $H_{k-1} + xH_{k-2}$ on both sides gives

$$H_k = (x+1)H_{k-1} - (x-y)H_{k-2} + xH_{k-2},$$
$$H_k = (x+1)H_{k-1} - (x-y)H_{k-2} - (H_{k-1} - xH_{k-2}).$$

But $H_{k-1} - xH_{k-2} = yH_{k-3}$, so,

$$H_k = (x+1)H_{k-1} - (x-y)H_{k-2} - yH_{k-3}.$$

That is,

$$H_{k+1} = (x+1)H_k - (x-y)H_{k-1} - yH_{k-2}.$$

Note that this matches equation (64). The adjustments in the last two rows of the transition matrix are based on the probabilistic properties of transition matrices (that is, each row must sum to 1). The entries in the last two columns are derived as follows:

Given the Horadam sequence $H(j, i, x, y)$ for the positive version, the first few terms of the resulting sequence are

$$j, i, xi + yj, x(xi + yj) + yi, x\{x(xi + yj) + yi\} + y(xi + yj), \ldots.$$

Since the limiting probability vector forms the normalized Horadam sequence (neglecting the first term), the resulting new sequence is also given by

$$i, xi + yj, x(xi + yj) + yi, x\{x(xi + yj) + yi\} + y(xi + yj), \ldots.$$

Then the following entries are derived as shown below.

(1) The $(n-1, n-1)$ entry of M is obtained by by inserting the ratio of the second entry of the new sequence to the first entry as the coefficient of b.

(2) The $(n-1, n-2)$ entry of the transition matrix is obtained by subtracting the fourth term of the sequence from the product of the third terms and $(x+1)$. That is,

$$(x+1)\{\frac{xi + yj}{i}\} - \frac{x(xi + yj) + yi}{i} = \frac{(x-y)i + yj}{i}.$$

(3) Since each row of the transition matrices sum to 1, the $(n-1, n-3)$ entry is obtained by subtracting the $(n-1, n-2)$ entry from $(n-1, n-1)$ entry.

$$\frac{xi + yj}{i} - \frac{(x-y)i + yj}{i} = \frac{yi}{i} = y.$$

Using this construction, the transition matrix has as limiting vector the normalized Horadam sequence. □

Moreover, it is observed that the above class of transition matrices has a limiting probability vector that does not form the normalized sequence of every number of the sub class of Horadam sequences given by

$$H_n = xH_{n-1} - yH_{n-2}. \tag{65}$$

Instead, we present another "nice" class of transition matrices whose limiting probability vector is the normalized sequence of the subclass of sequences in (65).

THEOREM 8.1.2.

For $0 < b \leq \frac{1}{x}$, $(xi - yj) > yi, i > 0$, *and* $x \geq y+1$, *the class of transition matrices of the form*

$$M = \begin{pmatrix} 1-b & b & 0 & 0 & . & . & . & . & 0 \\ (x-1)b & 1-xb & b & 0 & . & . & . & . & 0 \\ (x-y-1)b & yb & 1-xb & b & . & . & . & . & 0 \\ (x-y-1)b & 0 & yb & 1-xb & . & . & . & . & 0 \\ (x-y-1)b & 0 & 0 & yb & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & yb & 1-xb & b \\ \frac{xi-yj-yi}{i}b & 0 & . & . & . & . & . & yb & 1-\frac{xi-yj}{i}b \end{pmatrix}$$

has the limiting probability vector of the normalized sequence of every number of the sequence in (65).

PROOF.

Let the limiting probability vector be

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_{n-1}).$$

For $k = 1, 2, \ldots, n-2$, the balance equation for the above transition matrix is given by

$$\nu_k = b\nu_{k-1} + (1 - xb)\nu_k + yb\nu_{k+1},$$

that is,

$$x\nu_k = \nu_{k-1} + y\nu_{k+1}. \tag{66}$$

Using reversal we must show that

$$xH_k = H_{k+1} + yH_{k-1} \tag{67}$$

From the definition of the Horadam sequence numbers in (65),

$$H_k = xH_{k-1} - yH_{k-2}$$

This is seen to match equation (67) and in turn equation (67) is equivalent to (66), and the result is proved. □

It will be observed that this new class of transition matrices has the same pattern as the class of transition matrices presented in Theorem (7.2.1).

8.2. Queueing Models With Thresholds

For most classes of models considered so far, the limiting probabilities have been increasing (or decreasing) as the state number increases. In this section we consider another class of model formed where the limiting probabilities increase until a threshold is reached and then decrease after the threshold. We present an interesting class of transition matrices that models this type of situation.

THEOREM 8.2.1.

Let $M = x^2 + 2y$. Then, for $0 < b \leq \frac{1}{M}$ and $i > 0$, the class of $n \times n$ transition matrices of the form

$$\begin{pmatrix} 1-b & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-b & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-b & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-b & b & 0 & 0 \\ 0 & 0 & y^2b & y^2b & 1-Mb & b & 0 \\ 0 & y^2b & 0 & 0 & y^2b & 1-Mb & b \\ (\frac{x(xi\pm yj)\pm yi}{i} - y^2)b & 0 & 0 & 0 & 0 & y^2b & 1-\frac{x(xi\pm yj)\pm yi}{i}b \end{pmatrix}$$

has the limiting probability vector which is the normalized sequence of "every other number" of Horadam sequence. That is,

$$\nu = \frac{1}{\sum_{k=1}^{k=7} H_k}(H_2, H_4, H_6, H_7, H_5, H_3, H_1).$$

Extending this to $(2n+1)\times(2n+1)$ *transition matrices, the limiting probability vector has the general form*

$$\nu = \frac{1}{\sum_{k=1}^{k=2n+1} H_k}(H_2, H_4, ...H_{2n-2}, H_{2n}, H_{2n+1}, H_{2n-1}, H_{2n-3}, ..., H_3, H_1)$$

where the threshold is reached at state $(n+1)$*. For* $2n \times 2n$ *transition matrices, the limiting probability vector is given as*

$$\nu = \frac{1}{\sum_{k=1}^{k=2n} H_k}(H_2, H_4, ...H_{2n-2}, H_{2n}, H_{2n-1}, H_{2n-3}, ..., H_3, H_1)$$

and the threshold is reached at state n*.*

Real life processes that can be modelled by the above transition matrices abound in nature. For example, consider a patient that is ignorant of having cancer . Over time, the cancer cells build up in his system until he begins to experience symptoms (that is, when the threshold is reached). On visiting the doctor, the type of treatment prescribed determines the amount of cancer cells to be removed. The treatment options include drug administration or a major surgery. Administering drugs removes only a few cells at a time and the drop or increase depends on the degree of success. If a major surgery is recommended, infected tissue is removed and hence, a considerable amount of cancer cells are removed, so amjor drop occurs.

While Figure 8.1 shows the pattern observed in this class of transition matrices, Figure 8.2 is a graphical plot of the limiting probabilities in each state against the state number. The threshold of the graph corresponds to the "build up" in the system before the "service" starts.

Some of the peculiar features of this class of model are given below.

(1) The arrival rate into the system is constant.

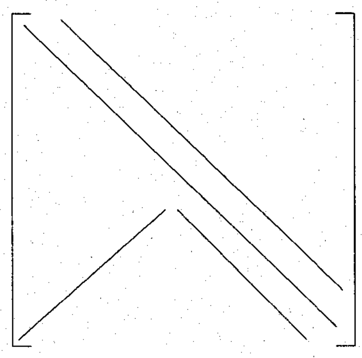


FIGURE 8.1. The observed matrix pattern.

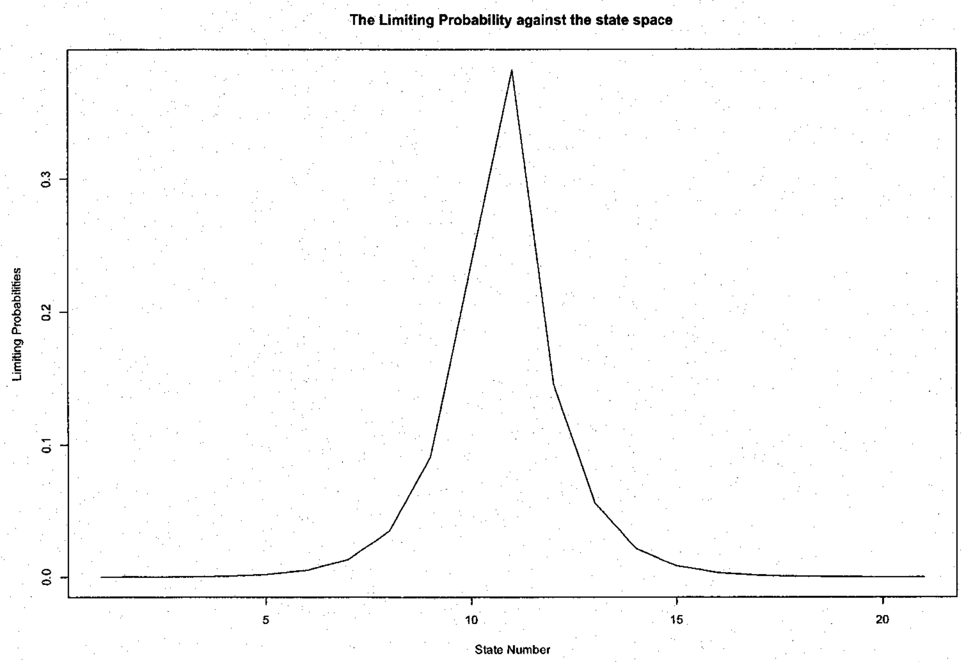


FIGURE 8.2. The graph of limiting probabilities against the state number.

(2) Service does not start until a threshold is reached.

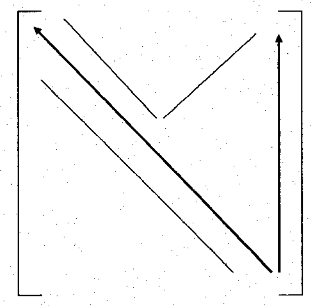


FIGURE 8.3. The observed transformed matrix pattern.

(3) The service pattern can be in singletons or in batches proportional to the state number.

(4) The rate of service for the individuals is equal to the service rate for the batches.

Of interest to system designers is the limiting probability vector (for infinite state space) of systems that can be modelled with the above transition matrix. It will be observed that this transition matrix does not allow an extension to an infinite state space.

8.3. Properties of the Transition Matrices

In this section we examine some common properties of classes of transition matrices considered thus far.

8.3.1. On Transformation of Transition Matrices.

For most of the classes of transition matrices considered so far in this thesis, the limiting probabilities decrease as the state increases. Moreover, as illustrated Theorem 3.3, a matrix can be transformed to get the transition matrix whose limiting probabilities increase as the state increases. It is observed that this approach can be extended to the other classes of transition matrices we considered to get a different system whose limiting probabilities increase as the state number increases.

As an example of a transformation, we consider the class of transition matrices in Theorem (8.2.1). After reflecting over the matrix center, we have a class of $n \times n$ transition matrices with the same pattern as the 7×7 transition matrix given below:

$$\begin{pmatrix} 1-\frac{x(xi\pm yj)\pm yi}{i}b & b & 0 & 0 & 0 & 0 & (\frac{x(xi\pm yj)\pm yi}{i}-y^2)b \\ b & 1-Mb & y^2b & 0 & 0 & y^2b & (M-2y^2-1)b \\ 0 & b & 1-Mb & y^2b & y^2b & 0 & (M-2y^2-1)b \\ 0 & 0 & b & 1-b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 1-b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 1-b & 0 \\ 0 & 0 & 0 & .. & .. & b & 1-b \end{pmatrix}.$$

The resulting limiting vector is given as

$$\nu = \frac{1}{\sum_{k=1}^{k=2n} H_k}(H_1, H_3, ...H_{2n-3}, H_{2n-1}, H_{2n+1}, H_{2n-2}, H_{2n-4}, ..., H_4, H_2).$$

The observed pattern in this class of transformed transition matrices is given in Figure 8.3.

8.3.2. On Uniqueness of the Classes of Transition Matrices.

One of the pertinent issues that arises in this study is the uniqueness of the class of transition matrices that produce the limiting probability vector whose elements form different types of sequences. It is observed that some classes of transition matrices, whose limiting probability vector is a normalized vector sequence of numbers, are not unique. For some sequences, there exist more than one class of transition matrices (representing different systems) whose limiting probability vector forms the normalized sequences belonging to the Horadam class of.

For example, consider the classes of transition matrices mentioned in Theorem 3.0.2 and 6.3.1. The two different classes of matrices produce the same limiting probability vector. Recall the transition matrix in chapter 4, which represents a model of queueing systems where the arrival and service pattern increases in a Fibonacci pattern. The limiting probability vector is found to be proportional to Fibonacci numbers. Another example is the sequence of Pell numbers. Based on the general results in the previous chapter, we get two different classes of transition matrices whose limiting probability vector gives the normalized sequence of Pell numbers.

CHAPTER 9

Conclusion and Future Research

In this thesis, we have illustrated that the Fibonacci numbers (and variants) make their appearance in the limiting probability vectors of some classes of natural transition matrices and infinitesimal generators. Also, some general results concerning the class of transition matrices whose limiting probability vector is the normalized sequences of Horadam sequences have been presented. Certain properties of these classes of transition matrices have also been presented.

Traditionally, rate matrices for birth and death processes have been a major focus of probability models and will continue in that role. By our knowledge of number sequences, we have been able to obtain the limiting probability vectors for some classes of queueing models, beyond the birth and death processes. Also, our methods allow us to obtain limiting vectors for certain infinite state processes in a (perhaps) new and relatively easy manner, by working with properties of the finite state version. Hopefully, this work will encourage other models to be examined more carefully beyond the scope of this thesis.

Unanswered questions include the following. Can we obtain a "nice" general form of transition matrix such that the limiting probability vector gives every third Fibonacci number, every fourth Fibonacci number, and so on? Given an infinite state space transition matrix, under what circumstances can we finitely truncate (with adjustments) to get a sufficiently "nice" finite state limiting probability vector which can be used to obtain the infinite state limiting vector? Furthermore, while we have considered the class of sequences whose next term is the linear combination of the preceding two terms, can we construct interesting transition matrices from the knowledge of other sequences beyond the Horadam class of sequences? Are there other reasonable queueing models which can be described using the special matrix forms (just the nonzero positions) that occurred in this work?

Appendix A

THEOREM A 1. *For Fibonacci sequence numbers,*

(a) $lim_{n\to\infty}\frac{F_n}{F_{n-1}}$ *exists,*

(b) $lim_{n\to\infty}\frac{F_n}{\sum_{i=1}^{n} F_i}$ *exists.*

PROOF.

(a) The Fibonacci numbers can be written as

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

Hence

$$\frac{F_n}{F_{n-1}} = \frac{\alpha^n - \beta^n}{\alpha^{n-1} - \beta^{n-1}}.$$

Taking the limits as $n \to \infty$, $(\frac{\beta}{\alpha})^n \to 0$. Hence we have

$$lim_{n\to\infty}\frac{F_n}{F_{n-1}} = lim_{n\to\infty}\frac{\alpha - \beta(\frac{\beta}{\alpha})^n}{1-(\frac{\beta}{\alpha})^{n-1}}.$$

Therefore,

$$lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \alpha = \frac{1+\sqrt{5}}{2}.$$

(b) We know

$$F_n = F_{n-1} + F_{n-2} \quad n = 2, 3, \ldots .$$

Let $F_n = r^n$. Then, the corresponding characteristic equation is given as

$$r^2 = r^2 + r + 1. \qquad (68)$$

Solving for the above equation gives two distinct roots $r_1 = \frac{1+\sqrt{5}}{2}$, and $r_2 = \frac{1-\sqrt{5}}{2}$. Then F_n can be written as

$$F_n = a_1 r_1^n + a_2 r_2^n.$$

Then

$$\frac{F_n}{\sum_{i=1}^n F_n} = \frac{a_1 r_1^n + a_2 r_2^n}{a_1 \sum_{i=1}^n r_1^n + a_2 \sum_{i=1}^n r_2^n}.$$

Since r_1 is the largest root of the characteristic equation (68). The above can be simplified as

$$\frac{F_n}{\sum_{i=1}^n F_i} = \frac{a_1 {r_1}^n + a_2 r_2^n}{a_1 \sum_{i=1}^n (r_1^i) + a_2 \sum_{i=1}^n (r_2^i)}.$$

Dividing through by r_1^n and taking the limits, $lim_{n\to\infty}(\frac{r_2}{r_1})^n \to 0$. Thus we have

$$lim_{n\to\infty}\frac{F_n}{\sum_{i=1}^n F_i} = \frac{r_1 - 1}{r_1} = 1 - \ell.$$

□

THEOREM B 1. *For $x \geq y > 0$ in the class of Horadam sequences governed by*

$$H_n = xH_{n-1} + yH_{n-2} \quad n = 2, 3, \dots ,$$

(a) $lim_{n\to\infty}\frac{H_{n-1}}{H_n}$ *exists.*

(b) $lim_{n\to\infty}\frac{H_{n-1}}{\sum_{k=1}^n H_k}$ *exists.*

PROOF.

(a) For Horadam sequences governed by the above recursive relation, the resulting characteristic equation is given as

$$r^2 - xr - y = 0.$$

Solving for r gives

$$r_1 = \frac{x+\sqrt{x^2+4y}}{2}; \quad r_2 = \frac{x-\sqrt{x^2+4y}}{2}.$$

Assume the roots are distinct roots. Then,

$$H_n = xa_1{r_1}^n + ya_2{r_2}^n,$$

$$\frac{H_{n-1}}{H_n} = \frac{xa_1{r_1}^{n-1} + ya_2{r_2}^{n-1}}{xa_1{r_1}^n + ya_2{r_2}^n},$$

$$\frac{H_{n-1}}{H_n} = \frac{1+(\frac{a_2}{a_1})\frac{y}{x}(\frac{r_2}{r_1})^{n-1}}{r_1 + r_2(\frac{a_2}{a_1})\frac{y}{x}(\frac{r_2}{r_1})^{n-1}}.$$

Note that as $n \to \infty$,

$$(\frac{r_2}{r_1})^{n-1} = (\frac{x-\sqrt{x^2+4y}}{x+\sqrt{x^2+4y}})^{n-1} = (\frac{-4y}{(x+\sqrt{x^2+4y})^2})^{n-1} \to 0.$$

Therefore, $lim_{n\to\infty}\frac{H_{n-1}}{H_n} = \frac{1}{r_1}$ exists.

(b)

$$lim_{n\to\infty}\frac{H_{n-1}}{\sum_{k=1}^n H_k} = \frac{xa_1{r_1}^n + ya_2{r_2}^n}{xa_1\sum_{k=1}^n {r_1}^k + ya_2\sum_{k=1}^n r_2^k}.$$

This can be simplified to

$$lim_{n\to\infty}\frac{H_{n-1}}{\sum_{k=1}^n H_k} = \frac{1+(\frac{a_2}{a_1})\frac{y}{x}(\frac{r_2}{r_1})}{x\frac{r_1}{r_1-1} + \frac{y}{x}(\frac{a_2}{a_1})\frac{r_2}{r_2-1}(\frac{{r_2}^n-1}{{r_1}^n})}.$$

Therefore, as $n \to \infty$,

$$lim_{n\to\infty}\frac{H_{n-1}}{\sum_{k=1}^n H_k} = 1 - \frac{1}{r_1}$$

exists.

□

THEOREM C 1. *For $x \geq y+2 > 0$ in the class of Horadam sequences governed by*

$$H_n = xH_{n-1} - yH_{n-2} \quad n = 2, 3, \dots ,$$

(a) $lim_{n\to\infty}\frac{H_{n-1}}{H_n}$ *exists,*

(b) $lim_{n\to\infty}\frac{H_{n-1}}{\sum_{k=1}^{n} H_k}$ *exists.*

PROOF.

(a) For Horadam sequences governed by the above recursive relation, the resulting characteristic equation is given as

$$r^2 - xr + y = 0.$$

Solving for r gives

$$r_1 = \frac{x + \sqrt{x^2 - 4y}}{2}; \quad r_2 = \frac{x - \sqrt{x^2 - 4y}}{2}.$$

Assume, both roots are distinct, then

$$H_n = xa_1{r_1}^n - ya_2{r_2}^n,$$

$$\frac{H_{n-1}}{H_n} = \frac{xa_1{r_1}^{n-1} - ya_2{r_2}^{n-1}}{xa_1{r_1}^n - ya_2{r_2}^n},$$

$$\frac{H_{n-1}}{H_n} = \frac{1 - (\frac{a_2}{a_1})\frac{y}{x}(\frac{r_2}{r_1})^{n-1}}{r_1 - r_2(\frac{a_2}{a_1})\frac{y}{x}(\frac{r_2}{r_1})^{n-1}}.$$

Note that as $n \to \infty$,

$$(\frac{r_2}{r_1})^{n-1} = (\frac{x - \sqrt{x^2 - 4y}}{x + \sqrt{x^2 - 4y}})^{n-1} = (\frac{-4y}{(x + \sqrt{x^2 - 4y})^2})^{n-1} \to 0.$$

Therefore, $lim_{n\to\infty} \frac{H_{n-1}}{H_n} = \frac{1}{r_1}$ exists.

(b)

$$lim_{n\to\infty} \frac{H_{n-1}}{\sum_{k=1}^{n} H_k} = \frac{xa_1 {r_1}^n - ya_2 {r_2}^n}{xa_1 \sum_{k=1}^{n} {r_1}^k - ya_2 \sum_{k=1}^{n} r_2^k}.$$

This can be simplified to

$$lim_{n\to\infty} \frac{H_{n-1}}{\sum_{k=1}^{n} H_k} = \frac{1 - (\frac{a_2}{a_1}) \frac{y}{x} (\frac{r_2}{r_1})}{x \frac{r_1}{r_1 - 1} - \frac{y}{x} (\frac{a_2}{a_1}) \frac{r_2}{r_2 - 1} (\frac{{r_2}^n - 1}{{r_1}^n})}.$$

Therefore, as $n \to \infty$,

$$lim_{n\to\infty} \frac{H_{n-1}}{\sum_{k=1}^{n} H_k} = 1 - \frac{1}{r_1}$$

exists.

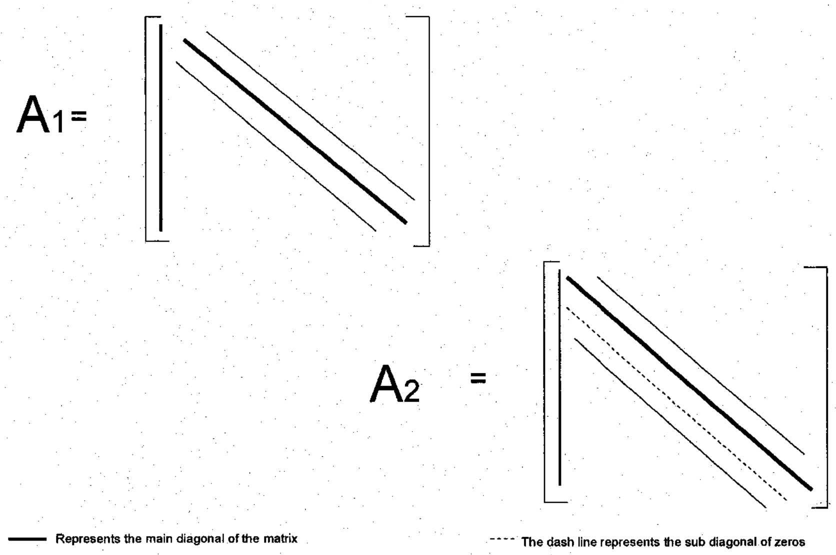
□

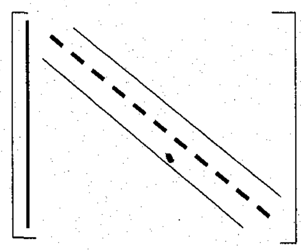
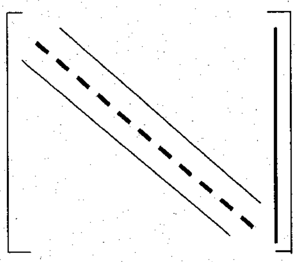
When the root is a double root, it can be confirmed that the limits still exist. When $x = k$ and $y = k - 1$, the limits exist and the results in Chapter 7 are valid.

Appendix B

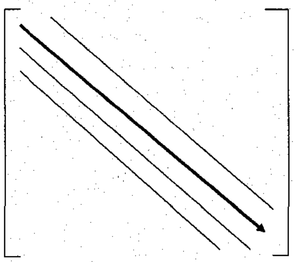
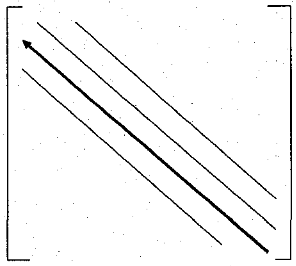
The following diagrams illustrate the patterns observed in different classes of transition matrices whose limiting probability vector is the normalized Horadam sequence numbers.

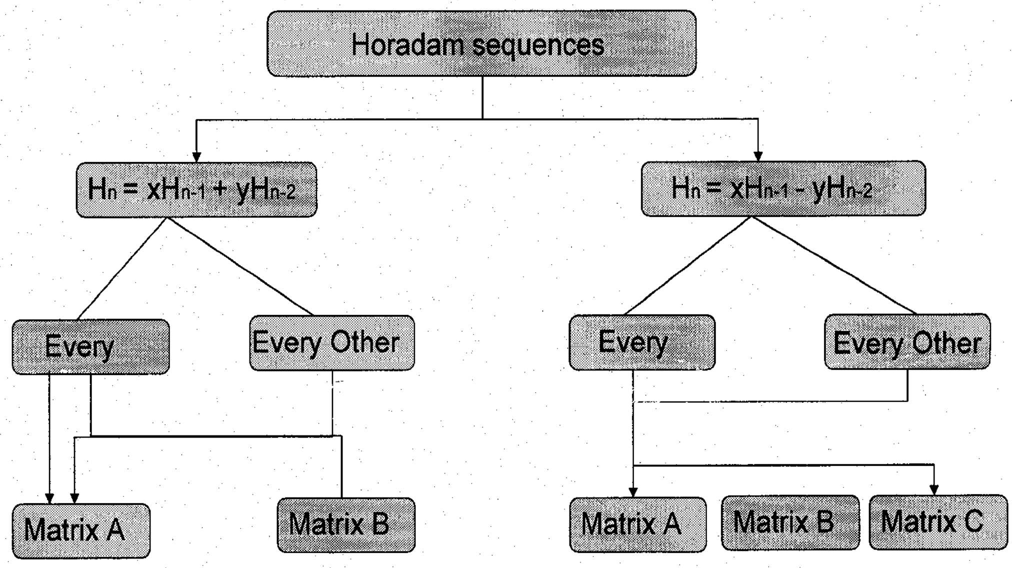
Observed Patterns for Classes of Transition Matrices



$C =$  $C_1 =$ 

– – Represents a different entry on the main diagonal of the matrix

$B =$  $B_1 =$ 



Pictorial Representation of the class of Transition Matrices

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